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ROCK BLOCKS, WREATH PRODUCTS AND KLR ALGEBRAS

ANTON EVSEEV

ABSTRACT. We consider RoCK (or Rouquier) blocks of symmetric groups and Hecke algebras at roots of unity. We prove a conjecture of Turner asserting that a certain idempotent truncation of a RoCK block of weight d of a symmetric group \mathfrak{S}_n defined over a field F of characteristic e is Morita equivalent to the principal block of the wreath product $\mathfrak{S}_e \wr \mathfrak{S}_d$. This generalises a theorem of Chuang and Kessar that applies to RoCK blocks with abelian defect groups. Our proof relies crucially on an isomorphism between $F\mathfrak{S}_n$ and a cyclotomic Khovanov–Lauda–Rouquier algebra, and the Morita equivalence we produce is that of graded algebras. We also prove the analogous result for an Iwahori–Hecke algebra at a root of unity defined over an arbitrary field.

1. INTRODUCTION

1.1. The main result. Let ξ be a fixed element of an arbitrary field F . We assume that there exists an integer $e \geq 2$ such that $1 + \xi + \cdots + \xi^{e-1} = 0$ and let e be the smallest such integer (the *quantum characteristic* of ξ). We fix e , F and ξ throughout the paper.

For an integral domain \mathcal{O} , an invertible element $\xi \in \mathcal{O}$ and an integer $n \geq 0$, the *Iwahori–Hecke algebra* $\mathcal{H}_n(\mathcal{O}, \xi)$ is the \mathcal{O} -algebra defined by the generators T_1, \dots, T_{n-1} subject to the relations

$$\begin{aligned} (1.1) \quad & (T_r - \xi)(T_r + 1) = 0 && \text{for } 1 \leq r < n, \\ (1.2) \quad & T_r T_{r+1} T_r = T_{r+1} T_r T_{r+1} && \text{for } 1 \leq r < n-1, \\ (1.3) \quad & T_r T_s = T_s T_r && \text{for } 1 \leq r, s < n \text{ such that } |r - s| > 1. \end{aligned}$$

Throughout, we write $\mathcal{H}_n = \mathcal{H}_n(F, \xi)$. The algebra \mathcal{H}_n is cellular, and hence F is necessarily a splitting field for this algebra (see e.g. [26, Theorem 3.20]).

It is well known that the blocks of \mathcal{H}_n are parameterised by the set

$$(1.4) \quad \text{Bl}_e(n) = \{(\rho, d) \in \text{Par} \times \mathbb{N} \mid \rho \text{ is an } e\text{-core and } |\rho| + ed = n\},$$

where Par is the set of all partitions. We write $b_{\rho, d}$ for the block idempotent of \mathcal{H}_n corresponding to $(\rho, d) \in \text{Bl}_e(n)$, and $\mathcal{H}_{\rho, d} = b_{\rho, d} \mathcal{H}_n$ denotes the corresponding block (see Section 2 for details). Representation theory of *RoCK* (or *Rouquier*) blocks of \mathcal{H}_n (see Definition 2.1) is much more tractable than that of blocks $\mathcal{H}_{\rho, d}$ in general. By a fundamental result of Chuang and Rouquier [8, Section 7], for any $d \geq 0$ and any two e -cores $\rho^{(1)}$ and $\rho^{(2)}$, the algebras $\mathcal{H}_{\rho^{(1)}, d}$ and $\mathcal{H}_{\rho^{(2)}, d}$ are derived equivalent. Consequently, in order to understand the structure of an arbitrary block $\mathcal{H}_{\rho, d}$ up to derived equivalence, it suffices to give a description of the structure of each RoCK block up to derived equivalence. If $\xi = 1$, then $e = \text{char } F$ is necessarily prime and $\mathcal{H}_n \cong F\mathfrak{S}_n$, where \mathfrak{S}_n denotes the symmetric group on n letters. Chuang and Kessar [6] proved that, when $\xi = 1$ and $d < \text{char } F = e$, a RoCK block $\mathcal{H}_{\rho, d}$ is Morita equivalent to the wreath product $\mathcal{H}_{\emptyset, 1} \wr \mathfrak{S}_d$. Note that here $\mathcal{H}_{\emptyset, 1}$ is the principal block of $F\mathfrak{S}_e$ and that the result of [6] applies precisely to RoCK blocks of symmetric groups with abelian defect. In fact, the aforementioned theorems of Chuang–Rouquier and Chuang–Kessar are stronger, as they hold with F replaced by an appropriate discrete valuation ring.

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When $d \geq \text{char } F$, the Morita equivalence of Chuang–Kessar no longer holds, as a RoCK block $\mathcal{H}_{\rho,d}$ has more isomorphism classes of simple modules than $\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d$. Nevertheless, Turner [33] conjectured in general (for $\xi = 1$) that $\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d$ is Morita equivalent to a certain idempotent truncation of a RoCK block. More precisely, for any integers $0 \leq m \leq n$, view \mathcal{H}_m as a subalgebra of \mathcal{H}_n via the embedding $T_j \mapsto T_j$ for $1 \leq j < m$. For any e -core ρ and $d \geq 0$, define

$$(1.5) \quad f_{\rho,d} = b_{\rho,0} b_{\rho,1} \cdots b_{\rho,d} \in \mathcal{H}_{|\rho|+de}.$$

Clearly, the factors in this product commute pairwise, so $f_{\rho,d}$ is an idempotent. The main result of this paper is the following theorem, which settles affirmatively [33, Conjecture 82] (stated in *loc. cit.* for the case $\xi = 1$).

Theorem 1.1. *Let $\mathcal{H}_{\rho,d}$ be a RoCK block and $f = f_{\rho,d}$. Then we have an algebra isomorphism $f\mathcal{H}_{\rho,d}f \cong \mathcal{H}_{\rho,0} \otimes_F (\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d)$. Hence, the algebra $f\mathcal{H}_{\rho,d}f$ is Morita equivalent to $\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d$.*

The second statement follows from the first one because $\mathcal{H}_{\rho,0}$ is a split simple algebra.

Remark 1.2. The formula defining the idempotent appearing in [33, Conjecture 82] is different to (1.5), but the resulting idempotent is equal to $f_{\rho,d}$: see Proposition 8.1.

While Theorem 1.1 is stated purely in the language of representation theory of symmetric groups and Hecke algebras at roots of unity, the proof given in this paper relies crucially on the fact that $\mathcal{H}_{\rho,d}$ is isomorphic to a certain cyclotomic Khovanov–Lauda–Rouquier (KLR) algebra. A consequence of this fact is that each of the algebras $\mathcal{H}_{\rho,d}$, $f_{\rho,d}\mathcal{H}_{\rho,d}f_{\rho,d}$ and $\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d$ has a natural \mathbb{Z} -grading. Moreover, $\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d$ is nonnegatively graded: this observation plays an important role in the proof. The isomorphism and the Morita equivalence in Theorem 1.1 are those of graded algebras (see Theorem 3.4 for a more precise statement).

In order to explain the meaning of Theorem 1.1 in more detail, we recall certain well-known general facts on idempotent truncation (see e.g. [15, Section 6.2]). Let A be an algebra over a field k and $\varepsilon \in A$ be an idempotent. Let $A\text{-mod}$ be the category of left modules over A . Then we have an exact functor $\mathcal{F}: A\text{-mod} \rightarrow \varepsilon A \varepsilon\text{-mod}$ defined as follows: for any A -module V , set $\mathcal{F}(V) = \varepsilon V$, and for any morphism $\phi: V \rightarrow W$ of A -modules, set $\mathcal{F}(\phi) = \phi|_{\varepsilon V}$. Further, the image $\mathcal{F}(D)$ of any simple A -module D is either simple or 0, and, if $\{D_\lambda \mid \lambda \in \Lambda\}$ is a complete and irredundant set of representatives of isomorphism classes of simple A -modules, then $\{\varepsilon D_\lambda \mid \lambda \in \Lambda, \varepsilon D_\lambda \neq 0\}$ is a complete and irredundant set of representatives of isomorphism classes of simple $\varepsilon A \varepsilon$ -modules. Informally, $\varepsilon A \varepsilon\text{-mod}$ captures the part of the structure of $A\text{-mod}$ that corresponds to the simple modules $D \in A\text{-mod}$ such that $\varepsilon D \neq 0$. In particular, if $\varepsilon D \neq 0$ for all simple A -modules D , then A is Morita equivalent to $\varepsilon A \varepsilon$.

When $A = \mathcal{H}_{\rho,d}$ is a RoCK block and $\varepsilon = f_{\rho,d}$, it is the case that $\varepsilon D \neq 0$ for all simple A -modules D if and only if $d < \text{char } F$ or $\text{char } F = 0$: see Proposition 8.2. Thus, when $d < \text{char } F$ or $\text{char } F = 0$, Theorem 1.1 yields a Morita equivalence between the RoCK block $\mathcal{H}_{\rho,d}$ and $\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d$. This equivalence was proved by Chuang and Miyachi [7, Theorem 18] under the assumption that either $\text{char } F = 0$ or ξ belongs to the prime subfield of F by using the aforementioned result of Chuang–Kessar (for $\xi = 1$) and similar results obtained independently by Miyachi [27] and Turner [32] for RoCK blocks of finite general linear groups (for $\xi \neq 1$). The proof given below is different from the arguments in the above papers. The isomorphism in Theorem 1.1 is constructed uniformly for all cases and is quite explicit, once the statement of the theorem is translated into the language of KLR algebras.

In the case when $\xi = 1$ and $e = 2$, Theorem 1.1 was proved by Turner (see [33, Theorem 84]) using a Brauer morphism. Independently of the present work, the same result was proved for $e = 2$ and arbitrary ξ by Konishi [23].

In the case when $\text{char } F = 0$, the decomposition matrix of a RoCK block $\mathcal{H}_{\rho,d}$ was determined by Chuang and Tan [9, Theorem 1.1] and, independently, by Leclerc and Miyachi [24, Corollary 10]. After a certain relabelling, this matrix may be seen to be identical to the decomposition matrix of $\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d$. When $\xi = 1$, the decomposition matrix of a RoCK block $\mathcal{H}_{\rho,d}$ was determined by Turner [33, Theorem 132] and was shown to coincide with that of $\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d$ by Paget [29, Theorem 3.4], again, after a certain relabelling. These results are closely related to Theorem 1.1 but are not directly implied by it, as we do not describe explicitly the $\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d$ -modules which are the images of Specht and simple modules of $\mathcal{H}_{\rho,d}$ under the composition of the Morita equivalence of Theorem 1.1 and the functor \mathcal{F} .

In addition to conjecturing the statement of Theorem 1.1, Turner [33] has constructed two remarkable algebras that he conjectured to be Morita equivalent respectively to the whole RoCK block $\mathcal{H}_{\rho,d}$ and to a RoCK block of a ξ -Schur algebra (see [33, Conjectures 165 and 178] respectively). After the present paper was submitted, the first of these conjectures was proved in [13] using results contained here.

1.2. Outline of the paper. Section 2 contains the definition of a RoCK block. In Section 3, we recall the definition of KLR algebras, state some of their standard properties and state a graded version of Theorem 1.1 as Theorem 3.4. The proof of Theorem 3.4 occupies Sections 4–7. A detailed outline of the proof is given in §3.3, after the required notation is introduced.

In Section 8, we prove two simple results that have already been referred to above and clarify Theorem 1.1. In Section 9, we give two alternative descriptions of the images in $f_{\rho,d}\mathcal{H}_{\rho,d}f_{\rho,d}$ of elementary transpositions of \mathfrak{S}_d under the isomorphism of Theorem 1.1, specifically,

- (i) an explicit formula for those images in terms of generators of the relevant KLR algebra (given without proof), see Equation (9.1);
- (ii) a formula in terms of the generators T_r of $\mathcal{H}_{|\rho|+de}$ and the grading on $f_{\rho,d}\mathcal{H}_{\rho,d}f_{\rho,d}$, see Proposition 9.5; in the case when $\xi = 1$, we give a simple description of the whole isomorphism in these terms, not just of its restriction to \mathfrak{S}_d : see Theorem 9.6.

These results provide different viewpoints on the isomorphism and may be useful for determining the images of simple and Specht modules of $\mathcal{H}_{\rho,d}$ under the Morita equivalence of Theorem 1.1 composed with the functor from $\mathcal{H}_{\rho,d}\text{-mod}$ to $f_{\rho,d}\mathcal{H}_{\rho,d}f_{\rho,d}\text{-mod}$ described above.

1.3. General notation. The symbol \mathbb{N} denotes the set of nonnegative integers. For integers $n > r > 0$, we denote by s_r the elementary transposition $(r, r+1) \in \mathfrak{S}_n$. If \mathcal{O} is a commutative ring, then \mathcal{O}^\times denotes the set of all invertible elements of \mathcal{O} . The centre of an algebra A is denoted by $Z(A)$. Throughout, subalgebras of F -algebras are not assumed to contain the identity element and F -algebra homomorphisms are not assumed to preserve the identity unless they are described as “unital”.

By a *graded* vector space (algebra, module) we mean a \mathbb{Z} -graded one. If V is a graded vector space, then $V_{\{n\}}$ denotes its n -th homogeneous component, so that $V = \bigoplus_{n \in \mathbb{Z}} V_{\{n\}}$. If $v \in V$, we write $v_{\{n\}}$ for the n -th component of v , so that $v_{\{n\}} \in V_{\{n\}}$ and $v = \sum_{n \in \mathbb{Z}} v_{\{n\}}$. For a subset $S \subset \mathbb{Z}$, we set $V_S = \bigoplus_{n \in S} V_{\{n\}} \leq V$ and $v_S = \sum_{n \in S} v_{\{n\}}$. We abbreviate $V_{<0}$ for $V_{\mathbb{Z}_{<0}}$, etc.

If the graded vector space V is finite-dimensional, then its graded dimension is defined as $\text{qdim } V = \sum_{n \in \mathbb{Z}} (\dim V_{\{n\}}) q^n \in \mathbb{Z}[q, q^{-1}]$. If A is a graded algebra and $m \in \mathbb{Z}$, then the graded algebra $A\langle m \rangle$ is defined to be the same algebra as A with the grading given by $A\langle m \rangle_{\{n\}} = A_{\{n-m\}}$ for all $n \in \mathbb{Z}$.

If U, V are F -vector spaces, we write $U \otimes V$ for $U \otimes_F V$. If X is a subset of an F -vector space, then FX denotes the F -span of X . If X and Y are vector subspaces of an algebra

A , we write $XY = F\{xy \mid x \in X, y \in Y\}$. For a symbol x , we often use the notation x^m as an abbreviation for x, \dots, x (m entries) or (x, \dots, x) (as appropriate). Also, we write $x^{\otimes m} = x \otimes \dots \otimes x$.

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2. RoCK BLOCKS

If $\lambda = (\lambda_1, \dots, \lambda_r)$ is a partition (so that $\lambda_1 \geq \dots \geq \lambda_r > 0$ are integers), we write $\ell(\lambda) = r$ and $|\lambda| = \sum_{j=1}^r \lambda_j$. Let \mathcal{O} be an integral domain and $t \in \mathcal{O}^\times$. For any partition λ of an integer n , let $S^{\lambda, \mathcal{O}, t}$ be the Specht $\mathcal{H}_n(\mathcal{O}, t)$ -module defined as in [26, Section 3.2]. We write $S^\lambda = S^{\lambda, F, \xi}$. Note that S^λ is the dual of the “Specht module” associated with λ constructed in [10].

For the definition of an e -core and e -weight of a partition, we refer the reader e.g. to [14, Chapter 2]. If $n \geq 0$ and $(\rho, d) \in \text{Bl}_e(n)$ (cf. (1.4)), then we define $b_{\rho, d} \in \mathcal{H}_n$ to be the unique block idempotent of \mathcal{H}_n such that $b_{\rho, d} S^\lambda = S^\lambda$ for all partitions λ of n with e -core ρ and e -weight d (see [11, Theorem 4.13]). The corresponding block algebra is $\mathcal{H}_{\rho, d} := b_{\rho, d} \mathcal{H}_n$.

We introduce notation related to abacus representations of partitions (see [14, Section 2.7]). Let λ be a partition and $l \geq 1$, $N \geq \ell(\lambda)$ be integers. We set $\lambda_r = 0$ for every integer $r > \ell(\lambda)$. The abacus display $\text{Ab}_N^l(\lambda)$ is the subset of $\mathbb{N} \times \{0, \dots, l-1\}$ defined by the property that $(t, i) \in \text{Ab}_N^l(\lambda)$ if and only if $lt + i \in \{\lambda_1 + N - 1, \lambda_2 + N - 2, \dots, \lambda_N\}$, whenever $t \geq 0$ and $0 \leq i < l$. The set $\mathbb{N} \times \{0, \dots, l-1\}$ is visualised as a table with infinitely many rows and l columns. The columns $0, \dots, l-1$ are drawn from left to right, and the entries $(0, i), (1, i), (2, i), \dots$ of each column are drawn from the top down. The columns $\mathbb{N} \times \{i\}$ are referred to as *runners*, and the elements of $\text{Ab}_N^l(\lambda)$ are referred to as *beads*. In particular, the number of beads of $\text{Ab}_N^l(\lambda)$ on runner i is defined as $|\text{Ab}_N^l(\lambda) \cap (\mathbb{N} \times \{i\})|$. We will write $\text{Ab}_N(\lambda) = \text{Ab}_N^e(\lambda)$, and we view $\text{Ab}_N^1(\lambda)$ as a subset of \mathbb{N} in the obvious way.

Definition 2.1. [33, Definition 52] *Let ρ be an e -core. We say that ρ is a Rouquier core for an integer $d \geq 1$ if there exists an integer $N \geq \ell(\rho)$ such that for all $i = 0, \dots, e-2$, the abacus display $\text{Ab}_N(\rho)$ has at least $d-1$ more beads on runner $i+1$ than on runner i . In this case, the block $\mathcal{H}_{\rho, d}$ is said to be a RoCK block.*

If ρ is a Rouquier core for some $d \geq 1$ and $N \geq \ell(\rho)$ is such that there is an abacus display as above, then $\kappa = -N + e\mathbb{Z} \in \mathbb{Z}/e\mathbb{Z}$ will be called a *residue* of ρ . It is easy to show that ρ has only one residue; in particular, this fact is a consequence of Lemma 4.3.

Example 2.2. Let $e = 3$ and $\rho = (8, 6, 4, 2, 2, 1, 1)$. The abacus display of ρ for $N = 7$ is

$$\begin{array}{ccc} \circ & \bullet & \bullet \\ \circ & \bullet & \bullet \\ \circ & \circ & \bullet \\ \circ & \circ & \bullet \\ \circ & \circ & \bullet \\ \circ & \circ & \circ \\ \circ & \circ & \circ \\ \vdots & & \end{array}$$

where the elements of the set $\text{Ab}_7(\rho)$ are represented by \bullet . Thus, $\mathcal{H}_{\rho, d}$ is a RoCK block of residue $2 + 3\mathbb{Z}$ for each $d = 1, 2, 3$.

3. KLR PRESENTATION OF \mathcal{H}_n

3.1. The KLR algebra and the Brundan–Kleshchev isomorphism. KLR algebras (also called quiver Hecke algebras) were introduced by Khovanov and Lauda [18] and independently by Rouquier [30]. We follow the presentation given in [2].

Let $I = \mathbb{Z}/e\mathbb{Z} = \{0, 1, \dots, e-1\}$. If $\mathbf{i} \in I^n = I \times \dots \times I$ for some $n \geq 0$, then i_r denotes the r -th entry of \mathbf{i} (for $1 \leq r \leq n$). Usually, we will write i_r instead of i_r and will use a similar convention for other bold symbols. If $\mathbf{i} \in I^n$ and $\mathbf{j} \in I^m$, we denote by \mathbf{ij} the concatenation $(i_1, \dots, i_n, j_1, \dots, j_m)$ of \mathbf{i} and \mathbf{j} .

Consider the quiver Γ with vertex set I , a directed edge from i to $i+1$ for each $i \in I$ and no other edges. Write $i \rightarrow j$ if there is an edge from i to j but not from j to i , $i \rightleftharpoons j$ if there are edges between i and j in both directions, and $i \nrightarrow j$ if $j \neq i, i \pm 1$. Let $\mathbf{C} = (c_{ij})_{i,j \in I}$ be the corresponding generalized Cartan matrix (of type $A_{e-1}^{(1)}$):

$$c_{ij} = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } i \nrightarrow j, \\ -1 & \text{if } i \rightarrow j \text{ or } i \leftarrow j, \\ -2 & \text{if } i \rightleftharpoons j. \end{cases}$$

For $i, j \in I$, define polynomials $L_{ij} \in F[y, y']$ by

$$L_{ij}(y, y') = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i \nrightarrow j, \\ y' - y & \text{if } i \rightarrow j, \\ y - y' & \text{if } i \leftarrow j, \\ (y' - y)(y - y') & \text{if } i \rightleftharpoons j. \end{cases}$$

The symmetric group \mathfrak{S}_n acts on I^n as follows: $w(i_1, \dots, i_n) = (i_{w^{-1}(1)}, \dots, i_{w^{-1}(n)})$ for $w \in \mathfrak{S}_n$. The *KLR algebra* R_n is the F -algebra generated by the set

$$(3.1) \quad \{e(\mathbf{i}) \mid \mathbf{i} \in I^n\} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}$$

subject to the relations

$$(3.2) \quad e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i}, \mathbf{j}}e(\mathbf{i}),$$

$$(3.3) \quad \sum_{\mathbf{i} \in I^n} e(\mathbf{i}) = 1,$$

$$(3.4) \quad y_r e(\mathbf{i}) = e(\mathbf{i}) y_r,$$

$$(3.5) \quad \psi_r e(\mathbf{i}) = e(s_r \mathbf{i}) \psi_r,$$

$$(3.6) \quad y_r y_s = y_s y_r,$$

$$(3.7) \quad \psi_r \psi_s = \psi_s \psi_r \quad \text{if } |r - s| > 1,$$

$$(3.8) \quad \psi_r y_s = y_s \psi_r \quad \text{if } s \neq r, r+1,$$

$$(3.9) \quad \psi_r y_{r+1} e(\mathbf{i}) = \begin{cases} (y_r \psi_r + 1) e(\mathbf{i}) & \text{if } i_r = i_{r+1}, \\ y_r \psi_r e(\mathbf{i}) & \text{if } i_r \neq i_{r+1}; \end{cases}$$

$$(3.10) \quad y_{r+1} \psi_r e(\mathbf{i}) = \begin{cases} (\psi_r y_r + 1) e(\mathbf{i}) & \text{if } i_r = i_{r+1}, \\ \psi_r y_r e(\mathbf{i}) & \text{if } i_r \neq i_{r+1}; \end{cases}$$

$$(3.11) \quad \psi_r^2 e(\mathbf{i}) = L_{i_r, i_{r+1}}(y_r, y_{r+1}) e(\mathbf{i}),$$

$$(3.12) \quad \psi_r \psi_{r+1} \psi_r e(\mathbf{i}) = \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} + 1)e(\mathbf{i}) & \text{if } i_{r+2} = i_r \rightarrow i_{r+1}, \\ (\psi_{r+1} \psi_r \psi_{r+1} - 1)e(\mathbf{i}) & \text{if } i_{r+2} = i_r \leftarrow i_{r+1}, \\ (\psi_{r+1} \psi_r \psi_{r+1} - 2y_{r+1} + y_r + y_{r+2})e(\mathbf{i}) & \text{if } i_{r+2} = i_r \rightleftharpoons i_{r+1}, \\ \psi_{r+1} \psi_r \psi_{r+1} e(\mathbf{i}) & \text{otherwise} \end{cases}$$

for all $\mathbf{i}, \mathbf{j} \in I^n$ and all admissible r and s .

Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of the Cartan matrix \mathbf{C} (see [16, §1.1]), with simple roots $\{\alpha_i \mid i \in I\}$, simple coroots $\{\alpha_i^\vee \mid i \in I\}$ and fundamental dominant weights $\{\Lambda_i \mid i \in I\}$ satisfying $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $i, j \in I$. Let $P_+ = \bigoplus_{i \in I} \mathbb{N} \Lambda_i$ and $Q_+ = \bigoplus_{i \in I} \mathbb{N} \alpha_i$. If $\alpha = \sum_{i \in I} n_i \alpha_i \in Q_+$, write $\text{ht}(\alpha) = \sum_{i \in I} n_i$.

Let $\Lambda \in P_+$. The *cyclotomic KLR algebra* R_n^Λ is defined as the quotient of R_n by the 2-sided ideal generated by the set $\{y_1^{\langle \Lambda, \alpha_{i_1}^\vee \rangle} e(\mathbf{i}) \mid \mathbf{i} \in I^n\}$. It follows from the above relations that the algebras R_n and R_n^Λ are both graded by the following rules: $\deg(e(\mathbf{i})) = 0$, $\deg(\psi_r e(\mathbf{i})) = -\mathbf{c}_{i_r, i_{r+1}}$ and $\deg(y_t) = 2$ whenever $\mathbf{i} \in I^n$, $1 \leq r < n$ and $1 \leq t \leq n$ (see [2]).

Remark 3.1. Elements of R_n may be represented as linear combinations of diagrams described by Khovanov and Lauda [18]. While diagrams are not used explicitly in our proof, the reader may find it helpful to translate some of the assertions below into the language of diagrams.

For each $i \in I$, define $\hat{i} \in F$ by

$$(3.13) \quad \hat{i} = \begin{cases} i & \text{if } \xi = 1, \\ \xi^i & \text{if } \xi \neq 1. \end{cases}$$

Here, and in the sequel, if $\xi = 1$, then i is identified with an element of F via the embedding $I = \mathbb{Z}/e\mathbb{Z} \rightarrow F$; and if $\xi \neq 1$, then $\xi^i = \xi^{i'} \in F$, where $i' \in \mathbb{Z}$ is any representative of the coset i . Let $H_n^\Lambda = H_n^\Lambda(\xi)$ be the *cyclotomic Hecke algebra* with parameter ξ ; see [2]. That is, H_n^Λ is the F -algebra generated by the set $\{T_1, \dots, T_{n-1}, X_1, \dots, X_n\}$ if $\xi = 1$ and by the set $\{T_1, \dots, T_{n-1}, X_1^{\pm 1}, \dots, X_n^{\pm 1}\}$ if $\xi \neq 1$ subject to the relations (1.1)–(1.3), the relation $\prod_{i \in I} (X_1 - \hat{i})^{\langle \Lambda, \alpha_i^\vee \rangle} = 0$ and the following relations:

- (a) if $\xi = 1$: $X_l X_t = X_t X_l$, $X_{r+1} = T_r X_r T_r + T_r$ and, if $t \notin \{r, r+1\}$, $X_t T_r = T_r X_t$;
- (b) if $\xi \neq 1$: $X_t^{\pm 1} X_l^{\pm 1} = X_l^{\pm 1} X_t^{\pm 1}$, $X_t X_t^{-1} = 1 = X_t^{-1} X_t$, $X_{r+1} = \xi^{-1} T_r X_r T_r$ and, if $t \notin \{r, r+1\}$, $T_r X_t = X_t T_r$,

for $1 \leq r < n$, $1 \leq t, l \leq n$. In particular, $H_n^{\Lambda_0}$ is isomorphic to \mathcal{H}_n via the map given by $T_r \mapsto T_r$ for $1 \leq r < n$ and $X_1 \mapsto \hat{0}$. In the sequel, we identify these two algebras.

Brundan and Kleshchev [2, Main Theorem] and independently Rouquier [30, Corollary 3.17] proved that the algebra H_n^Λ is isomorphic to R_n^Λ . More precisely, we have the following result.

Theorem 3.2 (Brundan–Kleshchev). *Let y and y' be indeterminates. There exist power series $P_i, Q_i \in F[[y, y']]$, $i \in I$, such that Q_i is invertible for each i and:*

(i) *For each $n \geq 0$ and $\Lambda \in P_+$, there is an isomorphism $\text{BK}_n^\Lambda: H_n^\Lambda \xrightarrow{\sim} R_n^\Lambda$ given by*

$$(3.14) \quad \text{BK}_n^\Lambda(T_r) = \sum_{\mathbf{i} \in I^n} (\psi_r Q_{i_r - i_{r+1}}(y_r, y_{r+1}) - P_{i_r - i_{r+1}}(y_r, y_{r+1})) e(\mathbf{i}) \quad \text{and}$$

$$(3.15) \quad \text{BK}_n^\Lambda(X_t) = \begin{cases} \sum_{\mathbf{i} \in I^n} (y_r + i_r) e(\mathbf{i}) & \text{if } \xi = 1, \\ \sum_{\mathbf{i} \in I^n} \xi^{i_r} (1 - y_r) e(\mathbf{i}) & \text{if } \xi \neq 1 \end{cases}$$

for $1 \leq r < n$ and $1 \leq t \leq n$.

- (ii) *For every $\mathbf{i} \in I^n$ and $t \in \{1, \dots, n\}$, the element $(\text{BK}_n^\Lambda(X_t) - \hat{i}_t) e(\mathbf{i})$ is nilpotent.*
- (iii) *If $\xi = 1$, then $P_i, Q_i \in F[[y - y']]$ for all $i \in I$.*

Proof. It is proved in [2, Sections 3 and 4] (see, in particular, [2, (3.41)–(3.42) and (4.42)–(4.43)]) that one has an isomorphism $H_n^\Lambda \xrightarrow{\sim} R_n^\Lambda$ given by

$$T_r \mapsto \sum_{\mathbf{i} \in I^n} (\psi_r Q'_{r,\mathbf{i}}(y_r, y_{r+1}) - P'_{r,\mathbf{i}}(y_r, y_{r+1})) e(\mathbf{i})$$

and (3.15) if power series $P'_{r,\mathbf{i}}, Q'_{r,\mathbf{i}} \in F[[y, y']]$, $1 \leq r < n$, $\mathbf{i} \in I^n$, satisfy certain explicit identities.¹ If $\xi = 1$, then the power series $P'_{r,\mathbf{i}}, Q'_{r,\mathbf{i}}$ given by [2, (3.22) and (3.30)] satisfy the required identities. Moreover, one easily checks that $P'_{r,\mathbf{i}} = P_{i_r - i_{r+1}}$ and $Q_{r,\mathbf{i}} = Q'_{i_r - i_{r+1}}$ for all r, \mathbf{i} provided the power series P_i and Q_i , $i \in I$, are defined as follows:

$$(3.16) \quad P_i = \begin{cases} 1 & \text{if } i = 0, \\ (i + y - y')^{-1} & \text{if } i \neq 0, \end{cases}$$

$$(3.17) \quad Q_i = \begin{cases} 1 + y' - y & \text{if } i = 0, \\ 1 - P_i & \text{if } i \notin \{0, 1, -1\}, \\ (1 - P_i^2)/(y' - y) & \text{if } e \neq 2 \text{ and } i = -1, \\ 1 & \text{if } e \neq 2 \text{ and } i = 1, \\ (1 - P_i)/(y' - y) & \text{if } e = 2 \text{ and } i = 1. \end{cases}$$

Since $P_i, Q_i \in F[[y - y']]$ for all i , this proves (i) when $\xi = 1$ and (iii).

Assuming that $\xi \neq 1$, let

$$(3.18) \quad P_i = \begin{cases} 1 & \text{if } i = 0, \\ (1 - \xi)(1 - \xi^i(1 - y)(1 - y')^{-1})^{-1} & \text{if } i \neq 0, \end{cases}$$

$$(3.19) \quad Q_i = \begin{cases} 1 - \xi + \xi y' - y & \text{if } i = 0, \\ \frac{\xi^i(1-y) - \xi(1-y')}{\xi^i(1-y) - (1-y')} & \text{if } i \notin \{0, 1, -1\}, \\ \frac{\xi^{-1}(1-y) - \xi(1-y')}{(\xi^{-1}(1-y) - (1-y'))^2} & \text{if } e \neq 2 \text{ and } i = -1, \\ 1 & \text{if } e \neq 2 \text{ and } i = 1, \\ \frac{1}{\xi(1-y') - (1-y)} & \text{if } e = 2 \text{ and } i = 1 \end{cases}$$

for all $i \in I$. (The only difference from power series given by [2, (4.27) and (4.36)] is a slight one in the formulas for Q_i in the cases when $i \in \{1, -1\}$.) As in [2], one checks that the power series $P'_{r,\mathbf{i}} = P_{i_r - i_{r+1}}$ and $Q_{r,\mathbf{i}} = Q_{i_r - i_{r+1}}$ satisfy the required properties, i.e. the identities [2, (4.27) and (4.33)–(4.35)], so one has an isomorphism given by (3.14)–(3.15).

Finally, (ii) follows from (3.15) and the fact that $y_t \in R_n^\Lambda$ is nilpotent for $1 \leq t \leq n$, as $\deg(y_t) = 2$ and $R_n^\Lambda \cong H_n^\Lambda$ is well known to be finite-dimensional. \square

From now on, we assume that BK_n^Λ is as in Theorem 3.2, with P_i and Q_i given by (3.16)–(3.19). We write

$$\text{BK}_n = \text{BK}_n^{\Lambda_0} : \mathcal{H}_n \xrightarrow{\sim} R_n^{\Lambda_0}.$$

For $0 \leq m \leq n$, we define a unital algebra homomorphism $\iota_m^n : R_m^{\Lambda_0} \rightarrow R_n^{\Lambda_0}$ by

$$(3.20) \quad e(\mathbf{i}) \mapsto \sum_{\mathbf{j} \in I^{n-m}} e(\mathbf{ij}), \quad \psi_r \mapsto \psi_r, \quad y_t \mapsto y_t$$

for $\mathbf{i} \in I^m$, $1 \leq r < m$ and $1 \leq t \leq m$. For every $\alpha \in Q_+$, define

$$I^\alpha = \{(i_1, \dots, i_n) \in I^n \mid \sum_{r=1}^n \alpha_{i_r} = \alpha\}$$

¹We write $P'_{r,\mathbf{i}}, Q'_{r,\mathbf{i}}$ for the power series denoted in [2] by $P_r(\mathbf{i}), Q_r(\mathbf{i})$ for $\xi \neq 1$ and by $p_r(\mathbf{i}), q_r(\mathbf{i})$ for $\xi = 1$.

and $e_\alpha = \sum_{\mathbf{i} \in I^\alpha} e(\mathbf{i})$, viewed either as an element of R_n or of $R_n^{\Lambda_0}$, depending on the context. The element e_α is a central idempotent. We write $R_\alpha = R_n e_\alpha$ and $R_\alpha^{\Lambda_0} = R_n^{\Lambda_0} e_\alpha$.

We identify every partition λ with its Young diagram, defined to be the set

$$\{(a, b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid 1 \leq a \leq \ell(\lambda), 1 \leq b \leq \lambda_a\}.$$

Whenever set-theoretic notation is used for a partition λ , it is to be viewed as a Young diagram. The residue of a box $(a, b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ is defined to be $\text{res}((a, b)) = b - a + e\mathbb{Z} \in I$, and the *residue content* of λ is defined as $\text{cont}(\lambda) = \sum_{(a,b) \in \lambda} \alpha_{\text{res}((a,b))} \in Q_+$. Note that if $\mu \subset \lambda$ are partitions and $\lambda \setminus \mu$ is an e -hook (i.e. a connected skew diagram of size e containing no 2×2 -squares), then each element of I occurs exactly once among the residues of the boxes of $\lambda \setminus \mu$. It follows that, if ρ is the e -core and d is the e -weight of λ , then $\text{cont}(\lambda) = \text{cont}(\rho) + d\delta$, where $\delta := \alpha_0 + \alpha_1 + \cdots + \alpha_{e-1} \in Q_+$ is the fundamental imaginary root.

Recall that, if $m \leq n$, then \mathcal{H}_m is viewed as a subalgebra of \mathcal{H}_n via $T_r \mapsto T_r$, $1 \leq r < m$. We state some standard properties of the isomorphism BK_n .

Proposition 3.3. *Let $n \geq 0$ be an integer.*

- (i) *For $0 \leq m \leq n$, we have $\iota_m^n \circ \text{BK}_m = \text{BK}_n|_{\mathcal{H}_m}$.*
- (ii) *For all $(\rho, d) \in \text{Bl}_e(n)$, we have $\text{BK}_n(b_{\rho,d}) = e_{\text{cont}(\rho)+d\delta}$. Hence, for every $(\rho, d) \in \text{Bl}_e(n)$, the map BK_n restricts to an algebra isomorphism from $\mathcal{H}_{\rho,d}$ onto $R_{\text{cont}(\rho)+d\delta}^{\Lambda_0}$.*

Proof. (i) easily follows from (3.14), and (ii) follows from [19, §2.9 and Theorem 5.6(ii)]. \square

Let $\mathcal{H}_{\rho,d}$ be a RoCK block of \mathcal{H}_n , and consider the idempotent $f_{\rho,d} \in \mathcal{H}_{\rho,d}$ defined by (1.5). By Proposition 3.3, we have

$$(3.21) \quad \text{BK}_n(f_{\rho,d}) = \prod_{r=0}^d \iota_{|\rho|+er}^{|\rho|+ed} (e_{\text{cont}(\rho)+r\delta}) = \sum_{\substack{\mathbf{j} \in I^{\text{cont}(\rho)} \\ \mathbf{i}^{(1)}, \dots, \mathbf{i}^{(d)} \in I^\delta}} e(\mathbf{j}\mathbf{i}^{(1)} \dots \mathbf{i}^{(d)}).$$

In particular, $\text{BK}_n(f_{\rho,d}) \in R_n^{\Lambda_0}$ is homogeneous of degree 0.

If A is a graded algebra over F , then the wreath product $A \wr \mathfrak{S}_d$ is defined as the algebra $A^{\otimes d} \otimes F\mathfrak{S}_d$ with multiplication given by

$$(x_1 \otimes \cdots \otimes x_d \otimes \sigma) \otimes (y_1 \otimes \cdots \otimes y_d \otimes \tau) = x_1 y_{\sigma^{-1}(1)} \otimes \cdots \otimes x_d y_{\sigma^{-1}(d)} \otimes \sigma\tau$$

for $x_1, \dots, x_d, y_1, \dots, y_d \in A$ and $\sigma, \tau \in \mathfrak{S}_d$. We identify $F\mathfrak{S}_d$ with a unital subalgebra of $A \wr \mathfrak{S}_d$ via the map $\sigma \mapsto 1^{\otimes d} \otimes \sigma$, $\sigma \in \mathfrak{S}_d$, and we identify $A^{\otimes d}$ with the unital subalgebra $A^{\otimes d} \otimes 1$ of $A \wr \mathfrak{S}_d$ in the obvious way. If A is graded, then we will view $A \wr \mathfrak{S}_d$ as a graded algebra via the rule

$$\deg(x_1 \otimes \cdots \otimes x_d \otimes \sigma) = \sum_{r=1}^d \deg(x_r)$$

whenever $x_1, \dots, x_d \in A$ are homogeneous and $\sigma \in \mathfrak{S}_r$.

Observe that the algebra R_δ is nonnegatively graded (that is, $(R_\delta)_{<0} = 0$): this follows from the fact that $e(\mathbf{i})e_\delta = 0$ if $\mathbf{i} \in I^e$ is such that $i_r = i_{r+1}$ for some r . Hence, the wreath product $R_\delta^{\Lambda_0} \wr \mathfrak{S}_d$ is also nonnegatively graded.

We will prove the following result, which may be viewed as a graded version of Theorem 1.1 and clearly implies that theorem (due to Proposition 3.3(ii)).

Theorem 3.4. *Let $n \geq 0$ and $(\rho, d) \in \text{Bl}_e(n)$ be such that ρ is a Rouquier core for d . If $f = \text{BK}_n(f_{\rho,d})$, then $fR_{\text{cont}(\rho)+d\delta}^{\Lambda_0}f$ and $R_{\text{cont}(\rho)}^{\Lambda_0} \otimes_F (R_\delta^{\Lambda_0} \wr \mathfrak{S}_d)$ are isomorphic as graded algebras.*

3.2. The standard basis and some general properties of R_α . Fix $\alpha \in Q_+$, and let $n = \text{ht}(\alpha)$. For $w \in \mathfrak{S}_n$, let $\ell(w)$ be the smallest m such that $w = s_{r_1} \cdots s_{r_m}$ for some $r_1, \dots, r_m \in \{1, \dots, n-1\}$. An expression $w = s_{r_1} \cdots s_{r_m}$ is said to be *reduced* if $m = \ell(w)$. We write $<$ and \leq for the Bruhat partial order on \mathfrak{S}_n . That is, $v \leq w$ if and only if there is a reduced expression $w = s_{r_1} \cdots s_{r_k}$ such that $v = s_{r_{a_1}} \cdots s_{r_{a_t}}$ for some a_1, \dots, a_t satisfying $1 \leq a_1 < \cdots < a_t \leq k$ (cf. e.g. [26, Section 3.1]).

If $w \in \mathfrak{S}_n$, we set $\psi_w = \psi_{r_1} \cdots \psi_{r_m} \in R_n$ where $w = s_{r_1} \cdots s_{r_m}$ is an arbitrary but fixed reduced expression for w . Note that, in general, ψ_w depends on the choice of a reduced expression. In the sequel, all results involving elements ψ_w are asserted to be true for *any* choice of reduced expressions used to construct ψ_w .

Theorem 3.5. (i) [18, Theorem 2.5] [30, Theorem 3.7] *The set*

$$\{\psi_w y_1^{m_1} \cdots y_n^{m_n} e(\mathbf{i}) \mid w \in \mathfrak{S}_n, m_1, \dots, m_n \in \mathbb{N}, \mathbf{i} \in I^\alpha\}$$

is a basis of R_α .

(ii) *Let $r_1, \dots, r_k \in \{1, \dots, n-1\}$, $w = s_{r_1} \cdots s_{r_k}$ and $g_0, \dots, g_k \in F[y_1, \dots, y_n]$. Then $g_0 \psi_{r_1} g_1 \psi_{r_2} \cdots g_{k-1} \psi_{r_k} g_k e_\alpha$ belongs to the span of elements of the form*

$$(3.22) \quad \psi_{r_{a_1}} \cdots \psi_{r_{a_l}} y_1^{m_1} \cdots y_n^{m_n} e(\mathbf{i})$$

where $1 \leq a_1 < \cdots < a_l \leq k$, $m_1, \dots, m_n \in \mathbb{N}$, $\mathbf{i} \in I^\alpha$ and the expression $s_{r_{a_1}} \cdots s_{r_{a_l}}$ is reduced.

Proof of (ii). Using relations (3.8)–(3.10) repeatedly, we see that $g_0 \psi_{r_1} g_1 \psi_{r_2} \cdots g_{k-1} \psi_{r_k} g_k e_\alpha$ belongs to the span of the elements (3.22) without the condition that $s_{r_{a_1}} \cdots s_{r_{a_l}}$ be reduced. Now the result follows from [5, Proposition 2.5]. \square

Let $\mu = (\mu_1, \dots, \mu_l)$ be a composition of n , i.e. a sequence of nonnegative integers such that $\sum_{r=1}^l \mu_r = n$. Let

$$\mathfrak{S}_\mu = \text{Aut}(\{1, \dots, \mu_1\}) \times \text{Aut}(\{\mu_1 + 1, \dots, \mu_1 + \mu_2\}) \times \cdots \cong \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_l},$$

a standard parabolic subgroup of \mathfrak{S}_n . Denote by \mathcal{D}_n^μ (respectively, ${}^\mu\mathcal{D}_n$) the set of the minimal length left (resp. right) coset representatives of \mathfrak{S}_μ in \mathfrak{S}_n . Note that an element $\sigma \in \mathfrak{S}_n$ belongs to \mathcal{D}_n^μ if and only if $\sigma(r) < \sigma(t)$ for all $r, t \in \{1, \dots, n\}$ such that $t \in \mathfrak{S}_\mu \cdot r$ and $r < t$; this fact and its analogue for ${}^\mu\mathcal{D}_n$ will be used repeatedly. If ν is another composition of n , set ${}^\nu\mathcal{D}_n^\mu = {}^\nu\mathcal{D}_n \cap \mathcal{D}_n^\mu$: this is the set of the minimal length double $(\mathfrak{S}_\nu, \mathfrak{S}_\mu)$ -coset representatives in \mathfrak{S}_n .

Let $\mu = (n_1, \dots, n_l)$ be a composition of n . There is an obvious map $\iota_\mu: R_{n_1} \otimes \cdots \otimes R_{n_l} \rightarrow R_n$, defined as the unique algebra homomorphism satisfying

$$\begin{aligned} \iota_\mu(e(\mathbf{i}^{(1)}) \otimes \cdots \otimes e(\mathbf{i}^{(l)})) &= e(\mathbf{i}^{(1)} \dots \mathbf{i}^{(l)}), \\ \iota_\mu(1^{\otimes r-1} \otimes \psi_k \otimes 1^{\otimes l-r}) &= \psi_{n_1 + \cdots + n_{r-1} + k}, \\ \iota_\mu(1^{\otimes r-1} \otimes y_t \otimes 1^{\otimes l-r}) &= y_{n_1 + \cdots + n_{r-1} + t} \end{aligned}$$

whenever $\mathbf{i}^{(1)} \in I^{n_1}, \dots, \mathbf{i}^{(l)} \in I^{n_l}$, $1 \leq r \leq l$, $1 \leq k < n_r$ and $1 \leq t \leq n_r$.

A *composition* of α is a tuple $(\gamma_1, \dots, \gamma_l)$ such that $\gamma_j \in Q_+$ for each j and $\sum_{j=1}^l \gamma_j = \alpha$. Let $\gamma = (\gamma_1, \dots, \gamma_l)$ be a composition of α , and set $\mu = \mu(\gamma) := (\text{ht}(\gamma_1), \dots, \text{ht}(\gamma_l))$. By restricting the map ι_μ to $R_{\gamma_1} \otimes \cdots \otimes R_{\gamma_l}$, we obtain an algebra homomorphism $\iota_\gamma: R_{\gamma_1} \otimes \cdots \otimes R_{\gamma_l} \rightarrow R_\alpha$. The image of this homomorphism will be denoted by R_γ and the image of $e_{\gamma_1} \otimes \cdots \otimes e_{\gamma_l}$ will be denoted by e_γ , so that $e_\gamma = e_{\gamma_1, \dots, \gamma_l}$ is the identity element of the subalgebra R_γ of R_α . If $w \in \mathfrak{S}_n$, then $w = w'v$ for some (uniquely determined) $w' \in \mathcal{D}_n^\mu$ and $v \in \mathfrak{S}_\mu$. We then have $\ell(w) = \ell(w') + \ell(v)$, and hence one can obtain a reduced expression for w by concatenating reduced expressions for w' and v . Using a reduced expression of this form to define each ψ_w , one deduces the following result from Theorem 3.5 (cf. the proof of [18, Proposition 2.16]).

Corollary 3.6. *For any composition γ of α , $R_\alpha e_\gamma$ is freely generated as a right R_γ -module by the set $\{\psi_w \mid w \in \mathcal{D}_n^{\mu(\gamma)}\}$.*

Proposition 3.7. *Let $\gamma = (\gamma_1, \dots, \gamma_l)$ and $\gamma' = (\gamma'_1, \dots, \gamma'_m)$ be compositions of α . Let $\mu = \mu(\gamma)$ and $\nu = \mu(\gamma')$. Then $e_{\gamma'} R_\alpha e_\gamma = \sum_{v \in {}^\nu \mathcal{D}_n^\mu} R_{\gamma'} \psi_v R_\gamma$.*

Proof. By Corollary 3.6, $e_{\gamma'} R_\alpha e_\gamma = \sum_{w \in \mathcal{D}_n^\mu} e_{\gamma'} \psi_w R_\gamma$. Let $w \in \mathcal{D}_n^\mu$, and let $u \in \mathfrak{S}_\nu$ and $v \in {}^\nu \mathcal{D}_n$ be such that $w = uv$ and $\ell(w) = \ell(u) + \ell(v)$. It is easy to show that $v \in {}^\nu \mathcal{D}_n^\mu$. We may assume that ψ_w is defined in such a way that $\psi_w = \psi_u \psi_v$ (for each w in question). Thus, $e_{\gamma'} \psi_w = e_{\gamma'} \psi_u \psi_v \in R_{\gamma'} \psi_v$, and the result follows. \square

Define R'_α to be the subalgebra of R_α generated by

$$\{e(\mathbf{i}) \mid \mathbf{i} \in I^\alpha\} \cup \{\psi_r e_\alpha \mid 1 \leq r < n\} \cup \{(y_r - y_t) e_\alpha \mid 1 \leq r, t \leq n\}.$$

The following fact was observed in [4, Lemma 3.1] (in a slightly different context). For the reader's convenience, we give a proof.

Proposition 3.8. *As a right $F[y_2 - y_1, y_3 - y_2, \dots, y_n - y_{n-1}]$ -module, R'_α is freely generated by the set $\{\psi_w e(\mathbf{i}) \mid w \in \mathfrak{S}_n, \mathbf{i} \in I^\alpha\}$.*

Proof. The fact that the given set generates a free right module U over $F[y_2 - y_1, \dots, y_n - y_{n-1}]$ follows immediately from Theorem 3.5. It remains only to show that $U = R'_\alpha$. Clearly, $U \subset R'_\alpha$.

Let z be an indeterminate, and consider $F[z] \otimes R_\alpha$, which is an $F[z]$ -algebra by extension of scalars, and hence an F -algebra. As is observed in [17, §1.3.2], there is an F -algebra homomorphism $\omega: R_n \rightarrow F[z] \otimes R_n$ given by

$$(3.23) \quad e(\mathbf{i}) \mapsto 1 \otimes e(\mathbf{i}), \quad \psi_r \mapsto 1 \otimes \psi_r, \quad y_t \mapsto z \otimes 1 + 1 \otimes y_t$$

for $\mathbf{i} \in I^n$, $1 \leq r < d$, $1 \leq t \leq d$. Note that $\omega(x) = 1 \otimes x$ for all $x \in R'_\alpha$. Also, it follows from Theorem 3.5 that R_α is freely generated as a left U -module by the set $\{y_1^j e_\alpha \mid j \geq 0\}$. Let $0 \neq x \in R'_\alpha$, and write $x = \sum_{j=0}^m u_j y_1^j$ where $u_j \in U$ for all j and $u_m \neq 0$. Then $1 \otimes x = \omega(x) \in z^m \otimes u_m + \sum_{j=0}^{m-1} z^j \otimes R_\alpha$, which forces $m = 0$. Hence, $x \in U$. \square

Let $\gamma = (\gamma_1, \dots, \gamma_l)$ be a composition of α . We define $R'_\gamma = R_\gamma \cap R'_\alpha$. Note that R'_γ need not be equal to $\iota_\gamma(R'_{\gamma_1} \otimes \dots \otimes R'_{\gamma_l})$.

Corollary 3.9. *As a right $F[y_2 - y_1, y_3 - y_2, \dots, y_n - y_{n-1}]$ -module, R'_γ is freely generated by the set $\{\psi_w e(\mathbf{i}) \mid w \in \mathfrak{S}_{\mu(\gamma)}, \mathbf{i} \in I^\alpha\}$.*

Proof. This is an immediate consequence of Corollary 3.6 and Proposition 3.8. \square

3.3. Outline of the proof of Theorem 3.4. We denote by $\text{Par}(n)$ the set of all partitions of n and by $\text{Par}_e(\rho, d)$ the set of all partitions with e -core ρ and e -weight d . A *standard tableau* of size n is a map $\mathbf{t}: \{1, \dots, n\} \rightarrow \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ such that \mathbf{t} is a bijection onto the Young diagram of a partition λ and the inverse of this bijection is increasing along the rows and columns of λ . In this situation, we say that λ is the *shape* of \mathbf{t} and write $\text{Shape}(\mathbf{t}) = \lambda$. We write $\text{Std}(\lambda)$ for the set of all standard tableaux of shape λ . The *residue sequence* of a standard tableau \mathbf{t} is defined as $\mathbf{i}^\mathbf{t} = (\text{res}(\mathbf{t}(1)), \dots, \text{res}(\mathbf{t}(n))) \in I^n$. For any e -core ρ and $d \geq 0$, define $I^{\rho, d}$ to be the set of all $\mathbf{i} \in I^n$ such that there exist $\lambda \in \text{Par}_e(\rho, d)$ and a standard tableau \mathbf{t} of shape λ satisfying $\mathbf{i}^\mathbf{t} = \mathbf{i}$.

Let (ρ, d) be an element of $\text{Bl}_e(n)$ (for some $n \geq 0$) such that ρ is a Rouquier e -core for d . Let κ be the residue of the RoCK block $\mathcal{H}_{\rho, d}$. As in the statement of Theorem 3.4, let $f = \text{BK}_n(f_{\rho, d})$. For $m \geq 0$, $\mathbf{i} \in I^m$ and $j \in I$, set $\mathbf{i}^{+j} = (i_1 + j, \dots, i_m + j)$. Let $I_{+j}^{\emptyset, 1} = \{\mathbf{i}^{+j} \mid \mathbf{i} \in I^{\emptyset, 1}\}$. Define the set

$$(3.24) \quad \mathcal{E}_{d, j} = \{w(\mathbf{i}^{(1)} \dots \mathbf{i}^{(d)}) \mid w \in \mathcal{D}_{ed}^{(e^d)}, \mathbf{i}^{(1)}, \dots, \mathbf{i}^{(d)} \in I_{+j}^{\emptyset, 1}\},$$

and set $\mathcal{E}_d = \mathcal{E}_{d,0}$. The following alternative description of $\mathcal{E}_{d,j}$ is verified easily: a tuple $\mathbf{i} \in I^{de}$ lies in $\mathcal{E}_{d,j}$ if and only if one can partition the set $\{1, \dots, de\}$ into subsets Y_1, \dots, Y_d of size e each such that for each $r = 1, \dots, d$, one has $(i_{a_1}, \dots, i_{a_e}) \in I_{+j}^{\emptyset,1}$ where $Y_r = \{a_1, \dots, a_e\}$ and $a_1 < \dots < a_e$.

Define $\bar{R}_{d\delta}$ to be the quotient of $R_{d\delta}$ by the 2-sided ideal generated by the set $\{e(\mathbf{i}) \mid \mathbf{i} \in I^{d\delta} \setminus \mathcal{E}_d\}$ and $\hat{R}_{d\delta}$ to be the quotient of $R_{d\delta}$ by the 2-sided ideal generated by $\{e(\mathbf{i}) \mid \mathbf{i} \in I^{d\delta} \setminus \mathcal{E}_{d,\kappa}\}$. It is clear from the definition of a KLR algebra that $R_{d\delta}$ has a graded automorphism given by

$$(3.25) \quad e(\mathbf{i}) \mapsto e(\mathbf{i}^{+\kappa}), \quad \psi_r e_{d\delta} \mapsto \psi_r e_{d\delta}, \quad y_t e_{d\delta} \mapsto y_t e_{d\delta}$$

whenever $\mathbf{i} \in I^{d\delta}$, $1 \leq r < de$ and $1 \leq t \leq de$. This automorphism corresponds to a rotational symmetry of the quiver Γ . Further, the map (3.25) clearly induces an isomorphism $\bar{R}_{d\delta} \xrightarrow{\sim} \hat{R}_{d\delta}$, which restricts to an isomorphism $\text{rot}_\kappa: e_{\delta d} \bar{R}_{d\delta} e_{\delta d} \xrightarrow{\sim} e_{\delta d} \hat{R}_{d\delta} e_{\delta d}$.

There is an obvious graded homomorphism $R_{d\delta} \rightarrow R_{\text{cont}(\rho)+d\delta}^{\Lambda_0}$, obtained as the composition of the natural projection $R_{\text{cont}(\rho)+d\delta} \rightarrow R_{\text{cont}(\rho)+d\delta}^{\Lambda_0}$ with the map $R_{d\delta} \rightarrow R_{\text{cont}(\rho)+d\delta}$, $x \mapsto \iota_{\text{cont}(\rho),d\delta}(e_{\text{cont}(\rho)} \otimes x)$. Using special combinatorial properties of RoCK blocks, we show in Section 4 that this map factors through $\hat{R}_{d\delta}$ and hence induces a graded algebra homomorphism $\Omega: e_{\delta d} \hat{R}_{d\delta} e_{\delta d} \rightarrow f R_{\text{cont}(\rho)+d\delta}^{\Lambda_0} f$. Further, the image $C_{\rho,d}$ of Ω has the property that $f R_{\text{cont}(\rho)+d\delta}^{\Lambda_0} f$ is isomorphic to $R_{\text{cont}(\rho)}^{\Lambda_0} \otimes C_{\rho,d}$ as a graded algebra (see Propositions 4.10 and 4.11). Thus, it is enough to show that $R_\delta^{\Lambda_0} \wr \mathfrak{S}_d \cong C_{\rho,d}$ as graded algebras.

In Section 5, we prove some elementary results on the structure of $R_\delta^{\Lambda_0}$, which are needed later. In Section 6, we construct a graded algebra homomorphism $\Theta: R_\delta^{\Lambda_0} \wr \mathfrak{S}_d \rightarrow \bar{R}_{d\delta}$. This allows us to define a homomorphism $\Xi: R_\delta^{\Lambda_0} \wr \mathfrak{S}_d \rightarrow C_{\rho,d}$ as the composition

$$(3.26) \quad R_\delta^{\Lambda_0} \wr \mathfrak{S}_d \xrightarrow{\Theta} e_{\delta d} \bar{R}_{d\delta} e_{\delta d} \xrightarrow{\text{rot}_\kappa} e_{\delta d} \hat{R}_{d\delta} e_{\delta d} \xrightarrow{\Omega} C_{\rho,d}.$$

In Section 7, we show that Ξ is surjective. Proposition 4.12 states that $R_\delta^{\Lambda_0} \wr \mathfrak{S}_d$ and $C_{\rho,d}$ have the same (graded) dimension, so we are then able to deduce that Ξ is an isomorphism, which concludes the proof.

The definition of the map Θ , unlike those of rot_κ and Ω , is far from straightforward. The crux of the proof is the construction in Section 6 of appropriate elements $\tau_r = \Theta(s_r) \in e_{\delta d} \bar{R}_{d\delta} e_{\delta d}$, where, as before, $s_r = (r, r+1) \in \mathfrak{S}_d \subset R_\delta^{\Lambda_0} \wr \mathfrak{S}_d$ for $1 \leq r < d$. In order to define τ_r and prove that they satisfy required relations, we adapt to the present context the ideas that Kang, Kashiwara and Kim [17] use to construct homomorphisms (“R-matrices”) between certain modules over KLR algebras.

The results of Section 6 are stated purely in the language of KLR algebras and do not involve a Rouquier core ρ . Intertwiners of the same flavour as τ_r appear to play an important role in representation theory of KLR algebras and were originally discovered (in a different context) by Kleshchev, Mathas, and Ram [20, Section 4]. More recently, module endomorphisms that are closely related to the elements τ_r have been constructed by Kleshchev and Muth (for KLR algebras of all untwisted affine types), also using the approach of [17]: see [21, Theorem 4.2.1].²

Remark 3.10. The main result of Section 6 is Theorem 6.14, which gives a partial description of the algebra $e_{\delta d} \bar{R}_{d\delta} e_{\delta d}$. A similar (but more explicit) result has independently been obtained by Kleshchev and Muth [22] for all KLR algebras of untwisted affine ADE types. More precisely, Theorem 6.14 can be deduced from [22, Theorem 5.9], which describes a

²For any fixed $\mathbf{i} \in I^{\emptyset,1}$, Kleshchev and Muth find a certain element of R_{de} , denoted by $\sigma_r + c$ in [21, (4.2.3)], such that the image of this element in $\bar{R}_{d\delta}$ multiplied by our $\Theta(e(\mathbf{i})^{\otimes d} \otimes 1)$ is equal to $\Theta(e(\mathbf{i})^{\otimes d} \otimes s_r)$.

certain idempotent truncation of $e_{\delta^d} \overline{R}_{d\delta} e_{\delta^d}$ as an affine zigzag algebra and is proved using explicit diagrammatic computations, which are generally avoided below. Also, Proposition 5.4 together with Lemma 6.5 are equivalent to the type A case of [22, Corollary 4.16].

4. COMBINATORICS OF A RoCK BLOCK

4.1. The algebra $C_{\rho,d}$. Let \mathbf{t} be a standard tableau of size $n \geq 0$. For any $m \leq n$, we write $\mathbf{t}_{\leq m} = \mathbf{t}|_{\{1, \dots, m\}}$. The degree $\deg(\mathbf{t})$ of \mathbf{t} is defined as follows (see [3, §4.11]). For $i \in I$, an i -node is a node $a \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ of residue i . Let μ be a partition. For a node $a \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, we say that a is an *addable* node for μ if $a \notin \mu$ and $\mu \cup \{a\}$ is the Young diagram of a partition, and we say that μ is a *removable* node of μ if $a \in \mu$ and $\mu \setminus \{a\}$ is the Young diagram of a partition. We say that a node $(r, t) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ is *below* a node (r', s') if $r > r'$. If a is an addable i -node of μ , define

$$(4.1) \quad d_a(\mu) = \#\{\text{addable } i\text{-nodes for } \mu \text{ below } a\} - \#\{\text{removable } i\text{-nodes of } \mu \text{ below } a\}.$$

Finally, define recursively

$$(4.2) \quad \deg(\mathbf{t}) = \begin{cases} d_{\mathbf{t}(n)}(\text{Shape}(\mathbf{t})) + \deg(\mathbf{t}_{\leq n-1}) & \text{if } n > 0, \\ 0 & \text{if } n = 0. \end{cases}$$

Recall the definition of the set $I^{\rho,d} \subset I^{\text{cont}(\rho)+d\delta}$ from §3.3 for any e -core ρ and $d \geq 0$.

Theorem 4.1. [3, Theorem 4.20] *For any integer $n \geq 0$ and $\mathbf{i}, \mathbf{j} \in I^n$, we have*

$$\text{qdim}(e(\mathbf{i})R_n^{\Lambda_0}e(\mathbf{j})) = \sum_{\substack{\lambda \in \text{Par}(n) \\ \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda) \\ \mathbf{i}^{\mathbf{s}} = \mathbf{i}, \mathbf{i}^{\mathbf{t}} = \mathbf{j}}} q^{\deg(\mathbf{s}) + \deg(\mathbf{t})}.$$

In particular, if $(\rho, d) \in \text{Bl}_e(n)$, then $e(\mathbf{i})e_{\text{cont}(\rho)+d\delta} \neq 0$ in $R_{\text{cont}(\rho)+d\delta}^{\Lambda_0}$ if and only if $\mathbf{i} \in I^{\rho,d}$.

The second assertion of the theorem follows from the first one because any two partitions with the same residue content have the same e -core, see [14, Theorem 2.7.41].

If X is a subset of $\mathbb{N} \times \{0, \dots, e-1\}$ and $a, b \in \mathbb{N} \times \{0, \dots, e-1\}$ are such that $a \in X$ and $b \notin X$, then we say that the set $(X \setminus \{a\}) \cup \{b\}$ is obtained from X by the move $a \rightarrow b$. If $(t, i) \in \mathbb{N} \times \{0, \dots, e-1\}$, then we say that the *next* node after (t, i) is the unique node $(t', j) \in \mathbb{N} \times \{0, \dots, e-1\}$ such that $et' + j = et + i + 1$ (i.e. $(t', j) = (t, i+1)$ if $i < e-1$ and $(t', j) = (t+1, 0)$ if $i = e-1$). We will use the following elementary fact.

Lemma 4.2. *Let $N \geq 0$ and λ be a partition such that $\ell(\lambda) \leq N$. Let $a \in \mathbb{N} \times \{0, \dots, e-1\}$ and $b = (t, i)$ be the next node after a . If $a \in \text{Ab}_N(\lambda)$, $b \notin \text{Ab}_N(\lambda)$ and $\text{Ab}_N(\mu)$ is obtained from $\text{Ab}_N(\lambda)$ by the move $a \rightarrow b$, then $\mu \setminus \lambda$ consists of a single node of residue $i - N + e\mathbb{Z}$.*

In the rest of this section, we assume that $\mathcal{H}_{\rho,d}$ is a RoCK block of residue κ and that $\text{Ab}_N(\rho)$ is an abacus display witnessing this fact. If X and Y are subsets of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ and there exists $c \in \mathbb{Z} \times \mathbb{Z}$ such that $Y = \{x + c \mid x \in X\}$, then we say that Y is a translate of X . The concept of two skew tableaux (viewed as maps $\{1, \dots, n\} \rightarrow \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$) being translates of each other is defined similarly. A partition of the form $(k, 1^{e-k})$ for some $k \in \{1, \dots, e\}$ will be called an e -hook partition. The following lemma includes a key combinatorial property of RoCK blocks, proved by Chuang and Kessar.

Lemma 4.3. *Let $1 \leq r < d$. Suppose that $\mu \in \text{Par}_e(\rho, r)$ and $\lambda \in \text{Par}_e(\rho, r+1)$ are such that $\mu \subset \lambda$. Then $\lambda \setminus \mu$ is the translate of a Young diagram of an e -hook partition. Moreover, the residue of the top-left corner of $\lambda \setminus \mu$ is equal to κ .*

Proof. The first statement is a part of [6, Lemma 4(2)]. By standard properties of the abacus (cf. [14, Section 2.7]), there exists $(t, i) \in \text{Ab}_N(\mu)$ such that $\text{Ab}_N(\lambda)$ is obtained from $\text{Ab}_N(\mu)$ by the move $(t, i) \rightarrow (t+1, i)$. By [6, Lemma 4(1)], $\{(t, i), (t, i+1), \dots, (t, e-1)\} \subset \text{Ab}_N(\mu)$ and $\{(t+1, 0), \dots, (t+1, i-1)\} \cap \text{Ab}_N(\mu) = \emptyset$. Hence, $\text{Ab}_N(\lambda)$ may be obtained from $\text{Ab}_N(\mu)$ by the following moves (in the given order), each of which corresponds to adding a single box to a Young diagram:

$$(4.3) \quad \begin{aligned} & (t, e-1) \rightarrow (t+1, 0), \\ & (t, e-2) \rightarrow (t, e-1), \dots, (t, i) \rightarrow (t, i+1), \\ & (t+1, 0) \rightarrow (t+1, 1), \dots, (t+1, i-1) \rightarrow (t+1, i). \end{aligned}$$

Hence, if ν denotes the partition such that $\nu \supset \mu$ and $\nu \setminus \mu$ consists of a single box which is the top-left corner of $\lambda \setminus \mu$, then $\text{Ab}_N(\nu)$ is obtained from $\text{Ab}_N(\mu)$ by the move $(t, e-1) \rightarrow (t+1, 0)$. By Lemma 4.2, the residue of the only box of $\nu \setminus \mu$ is $-N+e\mathbb{Z} = \kappa$. \square

Example 4.4. As in Example 2.2, let $e = 3$ and $\rho = (8, 6, 4, 2, 2, 1, 1)$, so that $\kappa = 2 + 3\mathbb{Z}$. Let $\nu = (8, 6, 4, 4, 3, 1, 1) \in \text{Par}_3(\rho, 1)$, $\mu = (11, 6, 4, 4, 3, 1, 1) \in \text{Par}_3(\rho, 2)$ and $\lambda = (11, 6, 4, 4, 3, 3, 2) \in \text{Par}_3(\rho, 3)$. Then $\rho \subset \nu \subset \mu \subset \lambda$, and each of $\nu \setminus \rho$, $\mu \setminus \nu$, $\lambda \setminus \mu$ is a translate of the Young diagram of a 3-hook partition. These translates are shown as hooks with thick boundaries in the following Young diagram of shape λ , which also gives the 3-residues of all boxes:

$$(4.4) \quad \begin{array}{ccccccccccc} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & \boxed{2} & \boxed{0} & \boxed{1} \\ 2 & 0 & 1 & 2 & 0 & 1 & & & & & \\ 1 & 2 & 0 & 1 & & & & & & & \\ 0 & 1 & \boxed{2} & \boxed{0} & & & & & & & \\ 2 & 0 & 1 & & & & & & & & \\ 1 & \boxed{2} & 0 & & & & & & & & \\ 0 & 1 & & & & & & & & & \end{array}$$

Let $j \in I$. The following lemma is an immediate consequence of the description of the set $\mathcal{E}_{d,j}$ given in §3.3.

Lemma 4.5. *Let $\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(d)} \in I^e$. If $\mathbf{i} = (\mathbf{i}^{(1)} \dots \mathbf{i}^{(d)}) \in \mathcal{E}_{d,j}$ and, for some $k > 0$, $\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(k)} \in I^\delta$, then $\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(k)} \in I_{+j}^{\emptyset, 1}$.*

Lemma 4.6. *If $\mathbf{j} \in I^{\rho, 0}$ and $\mathbf{i} \in I^{de}$ are such that $\mathbf{j}\mathbf{i} \in I^{\rho, d}$, then $\mathbf{i} \in \mathcal{E}_{d, \kappa}$.*

Proof. By the hypothesis, $\mathbf{j}\mathbf{i} = \mathbf{i}^\mathbf{t}$ for some $\lambda \in \text{Par}_e(\rho, d)$ and some standard tableau \mathbf{t} of shape λ . Since ρ is the e -core of λ , there is a sequence

$$\rho = \lambda^0 \subset \lambda^1 \subset \dots \subset \lambda^d = \lambda$$

of partitions such that $\lambda^r \in \text{Par}_e(\rho, r)$ for each $r = 0, \dots, d$. By Lemma 4.3, each $\lambda^r \setminus \lambda^{r-1}$ is a translate of an e -hook partition, and the top-left corner of $\lambda^r \setminus \lambda^{r-1}$ has residue κ . For $1 \leq r \leq d$, let $\mathbf{t}^{-1}(\lambda^r \setminus \lambda^{r-1}) = \{a_{r1}, \dots, a_{re}\}$, with $a_{r1} < \dots < a_{re}$. Since $\mathbf{j} \in I^{\rho, 0}$ and $\text{Par}_e(\rho, 0) = \{\rho\}$, we have $\mathbf{t}^{-1}(\rho) = \{1, \dots, |\rho|\}$. Hence, as \mathbf{t} is a standard tableau, $(i_{a_{r1}-|\rho|}, \dots, i_{a_{re}-|\rho|}) \in I_{+\kappa}^{\emptyset, 1}$. Therefore, the partition of $\{1, \dots, de\}$ into the subsets $\{a_{r1} - |\rho|, \dots, a_{re} - |\rho|\}$, $r = 1, \dots, d$, witnesses the fact that $\mathbf{i} \in \mathcal{E}_{d, \kappa}$. \square

Lemma 4.7. *For all $\mathbf{j} \in I^{\rho, 0}$, we have $\kappa \notin \{j_{\max\{1, |\rho|-e+1\}}, \dots, j_{|\rho|-1}, j_{|\rho|}\}$.*

The reader may find it helpful to check, by inspecting the residues in (4.4), that the lemma holds for e and ρ as in Example 4.4, i.e. that for any $\mathbf{i} \in I^{\rho, 0}$, none of the last 3 entries of \mathbf{i} is equal to 2.

Proof. Fix an integer $N \geq \ell(\rho)$ such that $-N + e\mathbb{Z} = \kappa$. Let \mathbf{t} be a standard tableau of shape ρ such that $\mathbf{i}^\mathbf{t} = \mathbf{j}$. Suppose for contradiction that $j_a = \kappa$ for some $a > |\rho| - e$, and choose such a to be largest possible. Let μ^r be the shape of $\mathbf{t}_{\leq a-1+r}$ for $r = 0, \dots, |\rho| - a + 1$. Then, by Lemma 4.2, $\mathbf{Ab}_N(\mu^1)$ is obtained from $\mathbf{Ab}_N(\mu^0)$ by the move $(t, e-1) \rightarrow (t+1, 0)$ for some $t \geq 0$. By maximality of a , the abacus $\mathbf{Ab}_N(\rho)$ can be obtained from $\mathbf{Ab}_N(\mu^1)$ by $|\rho| - a < e$ “horizontal” moves, i.e. moves of the form $(t', u) \rightarrow (t', u+1)$ for $t' \geq 0$, $0 \leq u < e-1$. Recall that, since ρ is a Rouquier core, for each $t' \in \mathbb{N}$ there exists $u \in \{0, \dots, e-1\}$ such that $\mathbf{Ab}_N(\rho) \cap (\{t'\} \times \{0, \dots, e-1\}) = \{(t', u), (t', u+1), \dots, (t', e-1)\}$, and the size of this intersection is weakly decreasing as t' increases. Let m be the number of beads in row t of $\mathbf{Ab}_N(\rho)$. Since $(t, e-1) \notin \mathbf{Ab}_N(\mu^1)$, at least m horizontal moves in row t are required to transform row t of $\mathbf{Ab}_N(\mu^1)$ to row t of $\mathbf{Ab}_N(\rho)$. Further, row $t+1$ of $\mathbf{Ab}_N(\rho)$ has at most m beads, so the leftmost bead of that row is in column numbered at least $e-m$. On the other hand, the leftmost bead of row t in $\mathbf{Ab}_N(\mu^1)$ is in column 0, so at least $e-m$ horizontal moves are required to transform row $t+1$ of $\mathbf{Ab}_N(\mu^1)$ into row $t+1$ of $\mathbf{Ab}_N(\rho)$. Hence, in total, at least e horizontal moves are needed to transform $\mathbf{Ab}_N(\mu^1)$ into $\mathbf{Ab}_N(\rho)$, which is a contradiction. \square

Combining Equation (3.21), Theorem 4.1 and Lemmas 4.5 and 4.6, we obtain the following formula:

$$(4.5) \quad \mathbf{BK}_{|\rho|+ed}(f_{\rho,d}) = \sum_{\substack{\mathbf{j} \in I^{\rho,0} \\ \mathbf{i}^{(1)}, \dots, \mathbf{i}^{(d)} \in I_{+\kappa}^{\emptyset,1}}} e(\mathbf{j}\mathbf{i}^{(1)} \dots \mathbf{i}^{(d)}).$$

If $\alpha \in Q_+$ and $\gamma = (\gamma_1, \dots, \gamma_l)$ is a composition of α , we define $R_\gamma^{\Lambda_0}$ to be the image of R_α under the natural projection $R_\alpha \rightarrow R_\alpha^{\Lambda_0}$.

Proposition 4.8. *If $f = \mathbf{BK}_{|\rho|+ed}(f_{\rho,d})$, then $fR_{\text{cont}(\rho)+d\delta}^{\Lambda_0}f \subset e_{\text{cont}(\rho),\delta^d}R_{\text{cont}(\rho),d\delta}^{\Lambda_0}e_{\text{cont}(\rho),\delta^d}$.*

Proof. Let $n = |\rho| + de$, $\mu = (|\rho|, e^d) = (|\rho|, e, \dots, e)$ and $\nu = (|\rho|, de)$. By (3.21), we have $f = e_{\text{cont}(\rho),\delta^d}$. Hence, by Proposition 3.7, $fR_{\text{cont}(\rho)+d\delta}^{\Lambda_0}f = \sum_{w \in {}^\mu\mathcal{D}_n^\mu} R_{\text{cont}(\rho),\delta^d}^{\Lambda_0} \psi_w R_{\text{cont}(\rho),\delta^d}^{\Lambda_0}$, so it will suffice to prove that $f\psi_w f = 0$ for all $w \in {}^\mu\mathcal{D}_n^\mu \setminus \mathfrak{S}_\nu$. Due to Equation (4.5) and relations (3.2) and (3.5), it is enough to show that for all such w we have $w(\mathbf{j}\mathbf{i}^{(1)} \dots \mathbf{i}^{(d)}) \neq \mathbf{j}'\mathbf{i}'^{(1)} \dots \mathbf{i}'^{(d)}$ whenever $\mathbf{j}, \mathbf{j}' \in I^{\rho,0}$ and $\mathbf{i}^{(r)}, \mathbf{i}'^{(r)} \in I_{+\kappa}^{\emptyset,1}$ for $r = 1, \dots, d$. Let $Y = \{1, \dots, |\rho|\}$ and $X_r = \{|\rho| + (r-1)e + 1, \dots, |\rho| + re\}$ for $r = 1, \dots, d$. Let $a \in \{1, \dots, n\}$ be maximal subject to $w(a) \in Y$. Since $w \notin \mathfrak{S}_\nu$, we have $a > |\rho|$. Let $X_r \ni a$ and $b = |\rho| + (r-1)e + 1$. Since $w \in {}^\mu\mathcal{D}_n^\mu$, we have $w(b) < w(a) \leq |\rho|$. Since $i_1^{(r)} = \kappa$, our assertion is true if $j'_{w(b)} \neq \kappa$, so we may assume that $j'_{w(b)} = \kappa$. By Lemma 4.7, this implies that $w(b) \leq |\rho| - e$. For each $c \in Z := \{w(b), w(b)+1, \dots, |\rho|\}$, we have $w^{-1}(c) \geq b$ because $w \in {}^\mu\mathcal{D}_n^\mu$ and $w^{-1}(c) \leq a$ by maximality of a . Since $|Z| > e$ and $\{b, b+1, \dots, a\} \subset X_r$, this is clearly impossible. \square

If V is a graded vector space, let $\text{END}(V)$ be the algebra of all endomorphisms of V (as an ungraded vector space), endowed with the unique grading such that $\deg(gv) = \deg(g) + \deg(v)$ for all homogeneous elements $g \in \text{END}(V)$ and $v \in V$.

Proposition 4.9. *Let λ be an e -core. Then there exists a graded vector space V such that $R_{\text{cont}(\lambda)}^{\Lambda_0} \cong \text{END}(V)$ as graded algebras.*

Proof. It is well known that $R_{\text{cont}(\lambda)}^{\Lambda_0} \cong \mathcal{H}_{\rho,0}$ is a split simple algebra (e.g. because it is a cellular algebra with only one cell, see [26, Corollary 5.38]), so $\mathcal{H}_{\rho,0} \cong \text{End}(V)$ as an ungraded algebra for some vector space V . By [28, Theorem 9.6.8], V can be graded as an $R_{\text{cont}(\rho)}^{\Lambda_0}$ -module, and the result follows. \square

If B is a subset and A is a subalgebra of an algebra A' , the centraliser of B in A is defined as $C_A(B) = \{a \in A \mid ab = ba \ \forall b \in B\}$. We will use the following elementary fact.

Proposition 4.10. *Let A be a finite-dimensional graded F -algebra. Suppose that B is a unital graded subalgebra of A such that $B \cong \text{END}(V)$ for some graded vector space V . Let $C = C_A(B)$. Then there is a graded algebra isomorphism $B \otimes C \xrightarrow{\sim} A$ given by $b \otimes c \mapsto bc$ for $b \in B$, $c \in C$. Moreover, for any homogeneous primitive idempotent ε of B , we have a graded algebra isomorphism $C \xrightarrow{\sim} \varepsilon A \varepsilon$ given by $c \mapsto \varepsilon c$.*

Proof. We view V as a B -module via the given isomorphism. Let $\{v_1, \dots, v_m\}$ be a homogeneous basis of V . For $1 \leq i, j \leq m$, let $e_{ij} \in B$ be the element given by $e_{ij}v_k = \delta_{jk}v_i$ for $k = 1, \dots, m$. Then $\{e_{ij} \mid 1 \leq i, j \leq m\}$ is a homogeneous basis of B , and $\{e_{ii} \mid 1 \leq i \leq m\}$ is a full set of primitive idempotents in B ; in particular, $\sum_{i=1}^m e_{ii} = 1$. Let $C' = e_{11}Ae_{11}$. It is straightforward to check that, for any $x \in C'$, the element $\xi(x) := \sum_{i=1}^m e_{i1}xe_{1i} \in A$ commutes with e_{jk} for $1 \leq j, k \leq m$, so $\xi(x) \in C$. It follows easily that the maps $\xi: C' \rightarrow C$ and $C \rightarrow C'$, $y \mapsto e_{11}y$, are mutually inverse isomorphisms of graded algebras. For any i and j , the graded vector space $e_{ii}Ae_{jj}$ is isomorphic to C' , as the maps $C' \rightarrow e_{ii}Ae_{jj}$, $x \mapsto e_{i1}xe_{1j}$ and $e_{ii}Ae_{jj} \rightarrow C'$, $y \mapsto e_{1i}ye_{j1}$ are mutual inverses. Observe also that for all $x \in C'$ we have $e_{ij}\xi(x) = e_{i1}xe_{1j}$ whenever $1 \leq i, j \leq m$. It follows that $e_{ij}C = e_{ii}Ae_{jj}$ for all i, j . Therefore, the graded algebra homomorphism defined in the statement of the proposition is an isomorphism $B \otimes C \xrightarrow{\sim} A$. The last statement has already been proved for $\varepsilon = e_{11}$ and follows in the general case because ε and e_{11} are conjugate by an invertible element of $B_{\{0\}}$ (both being primitive idempotents of $B_{\{0\}}$). \square

If $\alpha, \beta \in Q_+$ and $\text{ht}(\beta) = m \leq n = \text{ht}(\alpha)$, define a graded algebra homomorphism $\iota_{\beta}^{\alpha}: R_{\beta}^{\Lambda_0} \rightarrow R_{\alpha}^{\Lambda_0}$ by $x \mapsto e_{\alpha}\iota_m^n(x)$ (cf. (3.20)). As before, let

$$(4.6) \quad f = \text{BK}_{|\rho|+d\epsilon}(f_{\rho,d}) = e_{\text{cont}(\rho),\delta^d} \in R_{\text{cont}(\rho)+d\delta}^{\Lambda_0}.$$

Observe that f centralises $\iota_{\text{cont}(\rho)}^{\text{cont}(\rho)+d\delta}(R_{\text{cont}(\rho)}^{\Lambda_0})$ and that $f \neq 0$: the latter fact follows easily from (3.21) and Theorem 4.1. Hence, by Proposition 4.9, the map $R_{\text{cont}(\rho)}^{\Lambda_0} \rightarrow fR_{\text{cont}(\rho)+d\delta}^{\Lambda_0}f$ given by $x \mapsto \iota_{\text{cont}(\rho)}^{\text{cont}(\rho)+d\delta}(x)f$ is an injective unital graded algebra homomorphism, and its image is isomorphic to $\text{END}(V)$ for some graded vector space V . Therefore, defining

$$(4.7) \quad C_{\rho,d} = C_{fR_{\text{cont}(\rho)+d\delta}^{\Lambda_0}f}(\iota_{\text{cont}(\rho)}^{\text{cont}(\rho)+d\delta}(R_{\text{cont}(\rho)}^{\Lambda_0})),$$

we have a graded algebra isomorphism

$$(4.8) \quad R_{\text{cont}(\rho)}^{\Lambda_0} \otimes C_{\rho,d} \xrightarrow{\sim} fR_{\text{cont}(\rho)+d\delta}^{\Lambda_0}f$$

given by $a \otimes c \mapsto \iota_{\text{cont}(\rho)}^{\text{cont}(\rho)+d\delta}(a)c$ for $a \in R_{\text{cont}(\rho)}^{\Lambda_0}$ and $c \in C_{\rho,d}$, due to Proposition 4.10. Hence, in order to prove Theorem 3.4, it suffices to construct a graded isomorphism from $R_{\delta}^{\Lambda_0} \wr \mathfrak{S}_d$ onto $C_{\rho,d}$. Most of the remainder of the paper is devoted to this task.

Proposition 4.11. *Let $\omega: R_{d\delta} \rightarrow R_{\text{cont}(\rho),d\delta}^{\Lambda_0}$ be the graded algebra homomorphism defined as the composition $R_{d\delta} \rightarrow R_{\text{cont}(\rho),d\delta} \twoheadrightarrow R_{\text{cont}(\rho),d\delta}^{\Lambda_0}$ where the second map is the natural projection and the first one is given by $x \mapsto \iota_{\text{cont}(\rho),d\delta}(e_{\text{cont}(\rho)} \otimes x)$. Then:*

- (i) *We have $C_{\rho,d} = \omega(e_{\delta^d}R_{d\delta}e_{\delta^d})$.*
- (ii) *For any $\mathbf{i} \in I^{d\delta} \setminus \mathcal{E}_{d,\kappa}$, we have $\omega(e(\mathbf{i})) = 0$.*

Proof. It is clear from the definition that $\omega(e_{\delta^d}R_{d\delta}e_{\delta^d}) \subset e_{\text{cont}(\rho),\delta^d}R_{\text{cont}(\rho)+d\delta}^{\Lambda_0}e_{\text{cont}(\rho),\delta^d} = fR_{\text{cont}(\rho)+d\delta}f$ and that $\omega(R_{d\delta})$ commutes with $\iota_{\text{cont}(\rho)}^{\text{cont}(\rho)+d\delta}(R_{\text{cont}(\rho)}^{\Lambda_0})$. Thus, $\omega(e_{\delta^d}R_{d\delta}e_{\delta^d}) \subset$

$C_{\rho,d}$. For the converse, let $x \in C_{\rho,d}$. Then it follows from Proposition 4.8 that $x = \sum_{j=1}^m a_j c_j$ for some $a_1, \dots, a_m \in \iota_{\text{cont}(\rho)}^{\text{cont}(\rho)+d\delta}(R_{\text{cont}(\rho)}^{\Lambda_0})$ and $c_1, \dots, c_m \in \omega(e_{\delta d} R_{d\delta} e_{\delta d}) \subset C_{\rho,d}$, where a_1, \dots, a_m may be assumed to form a basis of $\iota_{\text{cont}(\rho)}^{\text{cont}(\rho)+d\delta}(R_{\text{cont}(\rho)}^{\Lambda_0})$, with $a_1 = f$. Due to injectivity of the map in Proposition 4.10, we infer that $x = c_1$, so $x \in \omega(e_{\delta d} R_{d\delta} e_{\delta d})$, and (i) is proved.

For (ii), note that for all $\mathbf{j} \in I^{\rho,0}$ we have $\mathbf{j}\mathbf{i} \notin I^{\rho,d}$ by Lemma 4.6 and hence $e(\mathbf{j}\mathbf{i}) = 0$ by Theorem 4.1. Thus, $\omega(e(\mathbf{i})) = \sum_{\mathbf{j} \in I^{\rho,0}} e(\mathbf{j}\mathbf{i}) = 0$. \square

Recall the definition of the quotient $\widehat{R}_{d\delta}$ of $R_{d\delta}$ in §3.3. By Proposition 4.11, the map ω defined in the statement of the proposition induces a homomorphism $\widehat{R}_{d\delta} \rightarrow R_{\text{cont}(\rho),d\delta}^{\Lambda_0}$, which restricts to a surjective graded algebra homomorphism $\Omega: e_{\delta d} \widehat{R}_{d\delta} e_{\delta d} \rightarrow C_{\rho,d}$.

4.2. The graded dimension of $C_{\rho,d}$. In this subsection, we prove the following result:

Proposition 4.12. *We have $\text{qdim}(C_{\rho,d}) = \text{qdim}(R_{\delta}^{\Lambda_0} \wr \mathfrak{S}_d) = d! \text{qdim}(R_{\delta}^{\Lambda_0})^d$. In particular, $C_{\rho,d}$ is nonnegatively graded.*

Turner ([33, Proposition 81]) proved the same result for ungraded dimensions in the case when $\xi = 1$. The proof given below is similar. If μ is an e -core and $a \geq 0$, let $\text{Std}_e(\mu, a)$ be the set of all standard tableaux with shape belonging to $\text{Par}_e(\mu, a)$. Let

$$\text{Std}'_e(\rho, d) = \{\mathbf{t} \in \text{Std}_e(\rho, d) \mid \mathbf{t}_{\leq |\rho|+rd} \in \text{Std}_e(\rho, r) \text{ for all } r = 0, \dots, d\}.$$

Define the map

$$\beta: \text{Std}'_e(\rho, d) \rightarrow \text{Std}_e(\rho) \times \text{Std}_e(\emptyset, 1)^{\times d}$$

by $\beta(\mathbf{t}) = (\mathbf{t}_{\leq |\rho|}, \mathbf{s}_1, \dots, \mathbf{s}_d)$ where, for $r = 1, \dots, d$, the tableau \mathbf{s}_r is the unique standard tableau which is a translate of the skew tableau of size e given by $m \mapsto \mathbf{t}(|\rho| + e(r-1) + m)$, $m = 1, \dots, e$; such a translate exists and has a shape belonging to $\text{Par}_e(\emptyset, 1)$ by Lemma 4.3.

For any $i \in \{0, \dots, e-1\}$, let v_i be the number of beads in the i -th column of $\text{Ab}_N(\rho)$. For any $\lambda \in \text{Par}_e(\rho, d)$, let $\lambda^{(i)}$ be the partition such that $\text{Ab}_{v_k}^1(\lambda^{(i)})$ is the projection onto the first component of $\text{Ab}_N(\lambda) \cap (\mathbb{N} \times \{i\})$. Up to a permutation, the sequence $(\lambda^{(0)}, \dots, \lambda^{(e-1)})$ is known as the e -quotient of λ .

Lemma 4.13. *Let $\lambda \in \text{Par}_e(\rho, d)$ and $(\mathbf{u}, \mathbf{s}_1, \dots, \mathbf{s}_d) \in \text{Std}(\rho) \times \text{Std}_e(\emptyset, 1)^{\times d}$. For each $i = 0, \dots, e-1$, let $d_i = \#\{r \in \{1, \dots, d\} \mid \text{Shape}(\mathbf{s}_r) = (i+1, 1^{e-i-1})\}$. Then*

$$|\beta^{-1}(\mathbf{u}, \mathbf{s}_1, \dots, \mathbf{s}_d) \cap \text{Std}(\lambda)| = \begin{cases} \prod_{i=0}^{e-1} |\text{Std}(\lambda^{(i)})| & \text{if } |\lambda^{(i)}| = d_i \text{ for } i = 0, \dots, e-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The map

$$(4.9) \quad \mathbf{t} \mapsto (\text{Shape}(\mathbf{t}_{\leq |\rho|}), \text{Shape}(\mathbf{t}_{\leq |\rho|+e}), \dots, \text{Shape}(\mathbf{t}_{\leq |\rho|+ed}))$$

is clearly a bijection from $\beta^{-1}(\mathbf{u}, \mathbf{s}_1, \dots, \mathbf{s}_d) \cap \text{Std}(\lambda)$ onto the set of sequences $\rho = \mu^0 \subset \mu^1 \subset \dots \subset \mu^d = \lambda$ of partitions such that $\mu^r \setminus \mu^{r-1}$ is a translate of $\text{Shape}(\mathbf{s}_r)$ for each $r = 1, \dots, d$. If $1 \leq r \leq d$, $0 \leq i \leq e-1$, $\nu \in \text{Par}_e(\rho, r-1)$ and $\mu \in \text{Par}_e(\rho, r)$ are such that $\nu \subset \mu$, then by Lemma 4.3 and [6, Lemma 4(2)], $\mu \setminus \nu$ is a translate of $(i+1, 1^{e-i-1})$ if and only if $\text{Ab}_N(\mu)$ is obtained from $\text{Ab}_N(\nu)$ by the move $(t, i) \rightarrow (t+1, i)$ for some $t \geq 0$. It follows immediately that $\beta^{-1}(\mathbf{u}, \mathbf{s}_1, \dots, \mathbf{s}_d) \cap \text{Std}(\lambda) = \emptyset$ unless $|\lambda^{(i)}| = d_i$ for all $i = 0, \dots, e-1$. Assuming that $|\lambda^{(i)}| = d_i$ for all i , for any given sequence $\rho = \mu^0 \subset \mu^1 \subset \dots \subset \mu^d = \lambda$ as above and any $i \in \{0, \dots, e-1\}$, let $\mathbf{t}_i \in \text{Std}(\lambda^{(i)})$ be defined as follows. Let $\{m_1 < \dots < m_{d_i}\}$ be the set of elements $m \in \{1, \dots, d\}$ such that $\text{Shape}(\mathbf{s}_m) = (i+1, 1^{e-i-1})$: then $\text{Shape}((\mathbf{t}_i)_{\leq k}) = (\mu^{m_k})^{(i)}$ for all $k = 1, \dots, d_i$. This assignment of a tuple $(\mathbf{t}_0, \dots, \mathbf{t}_{e-1})$ to each sequence $\rho = \mu^0 \subset \dots \subset \mu^d = \lambda$ with the above

properties defines a bijection from the set of such sequences onto $\text{Std}(\lambda^{(0)}) \times \cdots \times \text{Std}(\lambda^{(e-1)})$ and therefore, in view of (4.9), completes the proof. \square

Lemma 4.14. *For any $\mathbf{t} \in \text{Std}'_e(\rho, d)$, if $\beta(\mathbf{t}) = (\mathbf{u}, \mathbf{s}_1, \dots, \mathbf{s}_d)$, then $\deg(\mathbf{t}) = \deg(\mathbf{u}) + \sum_{r=1}^d \deg(\mathbf{s}_r)$.*

Proof. Due to (4.2), it is enough to prove that for any $r = 1, \dots, d$,

$$(4.10) \quad \sum_{k=1}^e d_{\mathbf{t}(|\rho|+e(r-1)+k)}(\text{Shape}(\mathbf{t}_{\leq |\rho|+e(r-1)+k})) = \deg(\mathbf{s}_r).$$

We will use the following general fact, which follows easily from (4.1). Let λ be a partition with $\ell(\lambda) \leq N$. For $t \geq 0$ and $0 \leq i < e$, write

$$c_{<t,i}(\lambda) = |\text{Ab}_N(\lambda) \cap \{(0, i), \dots, (t-1, i)\}|.$$

Let $a = (k, \lambda_k)$ be a removable node of λ , and let (t, i) be the bead of $\text{Ab}_N(\lambda)$ corresponding to this node, in the sense that $\lambda_k + N - k = et + i$. Then we have

$$(4.11) \quad d_a(\lambda) = \begin{cases} c_{<t,i-1}(\lambda) - c_{<t,i}(\lambda) & \text{if } i > 0, \\ c_{<t,e-1}(\lambda) - c_{<t+1,0}(\lambda) & \text{if } i = 0. \end{cases}$$

Let $\mu = \text{Shape}(\mathbf{t}_{\leq |\rho|+e(r-1)})$, $\nu = \text{Shape}(\mathbf{t}_{\leq |\rho|+er})$, and let $(t, i) \rightarrow (t+1, i)$ be the move converting $\text{Ab}_N(\mu)$ to $\text{Ab}_N(\nu)$. As in the proof of Lemma 4.3, we have $\{(t, i), (t, i+1), \dots, (t, e-1)\} \subset \text{Ab}_N(\mu)$ and $\{(t+1, 0), \dots, (t+1, i-1)\} \cap \text{Ab}_N(\mu) = \emptyset$. There exists an ordering M_1, \dots, M_e of the e moves listed under (4.3) such that, for each $k = 1, \dots, d$, $\text{Ab}_N(\text{Shape}(\mathbf{t}(|\rho| + e(r-1) + k)))$ is obtained from $\text{Ab}_N(\text{Shape}(\mathbf{t}(|\rho| + e(r-1) + k-1)))$ by the move M_k . Let $\mathbf{A} = \text{Ab}_N(\mu)$ and consider any abacus \mathbf{A}' obtained from \mathbf{A} by arbitrary addition or deletion of beads in any positions not belonging to the set $Z := \{(t, i), \dots, (t, e-1), (t+1, 0), \dots, (t+1, i)\}$. Let μ' be the partition defined by the condition that $\text{Ab}_{N'}(\mu') = \mathbf{A}'$, where N' is the number of beads in \mathbf{A}' , and set

$$(4.12) \quad d(\mathbf{A}') = \sum_{k=1}^e d_{a_k}(\mu' \cup \{a_1, \dots, a_k\}),$$

where a_1, \dots, a_e are the nodes added to μ' by the moves M_1, \dots, M_e in this order. In particular, $d(\mathbf{A})$ is the left-hand side of (4.10). We claim that $d(\mathbf{A}') = d(\mathbf{A})$. To prove this, it suffices to show that, for any \mathbf{A}' as above, $d(\mathbf{A}')$ does not change when one alters \mathbf{A}' by adding or deleting a bead in a position $(t', j) \notin Z$. If $t' > t+1$ or $t' = t+1$ and $j > i$, then a bead in position (t', j) does not affect the calculation of $d(\mathbf{A}')$ via the formula (4.11). On the other hand, if $t' < t$ or $t' = t$ and $j < i$, then the total contribution of any bead in position (t', j) to the calculation of $d(\mathbf{A}')$ via (4.11) is 0 (because such a bead contributes 1 to one of the summands of (4.12), -1 to another summand and 0 to the remaining summands). This proves the claim.

Now consider the abacus \mathbf{A}' obtained from \mathbf{A} by deleting all beads outside positions $(t, i), \dots, (t, e-1), (t+1, 0), \dots, (t+1, i)$ and then adding a bead in each of the positions $(t, 0), \dots, (t, i-1)$. Then $d(\mathbf{A}') = d(\mathbf{A})$. On the other hand, for each $k = 1, \dots, e$, the abacus obtained from \mathbf{A}' by the moves M_1, \dots, M_k is precisely the abacus $\text{Ab}_e((\mathbf{s}_r)_{\leq k})$ with t empty rows added on the top. It follows by (4.11) that $d(\mathbf{A}') = \deg(\mathbf{s}_r)$, and we have proved (4.10). \square

Proof of Proposition 4.12. The second equality in the statement is obvious, so we only need to prove that $\text{qdim}(C_{\rho,d}) = d! \text{qdim}(R_{\delta}^{\Lambda_0})^d$. For any e -core μ and $a \geq 0$, let

$$X_{\mu,a} = \{(\mathbf{t}, \mathbf{t}') \in \text{Std}_e(\mu, a) \mid \mathbf{t} \text{ and } \mathbf{t}' \text{ have the same shape}\}.$$

Using (4.6), Theorem 4.1 and Lemmas 4.13 and 4.14, we compute

$$\begin{aligned}
\text{qdim}(f R_{\text{cont}(\rho)}^{\Lambda_0} f) &= \sum_{(\mathbf{t}, \mathbf{t}') \in X_{\rho, d} \cap (\text{Std}'_e(\rho, d) \times \text{Std}'_e(\rho, d))} q^{\deg(\mathbf{t}) + \deg(\mathbf{t}')} \\
&= \sum_{\mathbf{u}, \mathbf{u}' \in \text{Std}(\rho)} q^{\deg(\mathbf{u}) + \deg(\mathbf{u}')} \sum_{\substack{d_0, \dots, d_{e-1} \geq 0 \\ d_0 + \dots + d_{e-1} = d}} \binom{d}{d_0, \dots, d_{e-1}}^2 \\
&\quad \times \prod_{i=0}^{e-1} \left(\sum_{\mathbf{s}_{i,1}, \dots, \mathbf{s}_{i,d_i}, \mathbf{s}'_{i,1}, \dots, \mathbf{s}'_{i,d_i} \in \text{Std}_e(i+1, 1^{e-i-1})} q^{\sum_{l=1}^{d_i} (\deg(\mathbf{s}_{i,l}) + \deg(\mathbf{s}'_{i,l}))} \sum_{\lambda^i \in \text{Par}(d_i)} |\text{Std}(\lambda^i)|^2 \right) \\
&= d! \sum_{\mathbf{u}, \mathbf{u}' \in \text{Std}(\rho)} q^{\deg(\mathbf{u}) + \deg(\mathbf{u}')} \sum_{\substack{d_0, \dots, d_{e-1} \geq 0 \\ d_0 + \dots + d_{e-1} = d}} \binom{d}{d_0, \dots, d_{e-1}} \\
&\quad \times \prod_{i=0}^{e-1} \left(\sum_{\mathbf{s}_{i,1}, \dots, \mathbf{s}_{i,d_i}, \mathbf{s}'_{i,1}, \dots, \mathbf{s}'_{i,d_i} \in \text{Std}_e(i+1, 1^{e-i-1})} q^{\sum_{l=1}^{d_i} (\deg(\mathbf{s}_{i,l}) + \deg(\mathbf{s}'_{i,l}))} \right) \\
&= d! \sum_{\mathbf{u}, \mathbf{u}' \in \text{Std}(\rho)} q^{\deg(\mathbf{u}) + \deg(\mathbf{u}')} \sum_{(\mathbf{s}_1, \mathbf{s}'_1), \dots, (\mathbf{s}_d, \mathbf{s}'_d) \in X_{\emptyset, 1}} q^{\sum_{r=1}^d (\deg(\mathbf{s}_r) + \deg(\mathbf{s}'_r))} \\
&= d! \text{qdim}(R_{\text{cont}(\rho)}^{\Lambda_0}) \text{qdim}(R_{\delta}^{\Lambda_0})^d,
\end{aligned}$$

where for the third equality we use the classical identity $\sum_{\mu \in \text{Par}(m)} |\text{Std}(\mu)|^2 = m!$, which holds for all $m \geq 0$. The desired identity is now obtained by dividing both sides by $\text{qdim}(R_{\text{cont}(\rho)}^{\Lambda_0})$ and using the graded isomorphism (4.8). \square

5. A HOMOGENEOUS BASIS OF $R_{\delta}^{\Lambda_0}$

Recall that $\delta = \alpha_0 + \dots + \alpha_{e-1} \in Q_+$ is the fundamental imaginary root. In R_{δ} , we have $\psi_r \psi_{r+1} \psi_r e_{\delta} = \psi_{r+1} \psi_r \psi_{r+1} e_{\delta}$ for all $r = 1, \dots, e-2$ because for each $\mathbf{i} \in I^{\delta}$ we have $i_t \neq i_{t'}$ for $1 \leq t < t' \leq e$. Hence, by Matsumoto's Theorem (see e.g. [26, Theorem 1.8]), for any $v \in \mathfrak{S}_e$, the element $\psi_v e_{\delta}$ does not depend on the choice of a reduced expression for v . For each $\mathbf{i}, \mathbf{j} \in I^{\delta}$, let $w_{\mathbf{i}, \mathbf{j}}$ be the unique element of \mathfrak{S}_e such that $w_{\mathbf{i}, \mathbf{j}} \mathbf{j} = \mathbf{i}$.

The following three lemmas are left as exercises for the reader. In the first two, we identify every $i \in I$ with the corresponding element of $\{0, 1, \dots, e-1\} \subset \mathbb{Z}$. If $\mathbf{i} \in I^n$, we say that a tuple \mathbf{j} is a *subsequence* of \mathbf{i} if $\mathbf{j} = (i_{a_1}, \dots, i_{a_m})$ for some $a_1, \dots, a_m \in \{1, \dots, n\}$ such that $a_1 < \dots < a_m$.

Lemma 5.1. *An element $\mathbf{i} \in I^{\delta}$ belongs to $I^{\emptyset, 1}$ if and only if $i_1 = 0$ and both $(1, 2, \dots, i_e - 1)$ and $(e-1, e-2, \dots, i_e + 1)$ are subsequences of \mathbf{i} .*

Lemma 5.2. *Let $\mathbf{i} \in I^{\emptyset, 1}$. Then the set $\{\mathbf{t} \in \text{Std}(e) \mid \mathbf{i}^{\mathbf{t}} = \mathbf{i}\}$ consists of precisely two tableaux, namely, a standard tableau of shape $(i_e + 1, 1^{e-i_e-1})$ and degree 1 and a standard tableau of shape $(i_e, 1^{e-i_e})$ and degree 0.*

Lemma 5.3. *If $0 \leq m < e$, then all standard tableaux of size m have degree 0.*

Proposition 5.4. *The algebra $R_{\delta}^{\Lambda_0}$ has a homogeneous basis $\mathcal{B}_0 \sqcup \mathcal{B}_1 \sqcup \mathcal{B}_2$ where*

$$\begin{aligned}
\mathcal{B}_0 &= \{\psi_{w_{\mathbf{j}, \mathbf{i}}} e(\mathbf{i}) \mid \mathbf{i}, \mathbf{j} \in I^{\emptyset, 1}, i_e = j_e\}, \\
\mathcal{B}_1 &= \{\psi_{w_{\mathbf{j}, \mathbf{i}}} e(\mathbf{i}) \mid \mathbf{i}, \mathbf{j} \in I^{\emptyset, 1}, i_e = j_e \pm 1\}, \\
\mathcal{B}_2 &= \{\psi_{w_{\mathbf{j}, \mathbf{i}}} y_e e(\mathbf{i}) \mid \mathbf{i}, \mathbf{j} \in I^{\emptyset, 1}, i_e = j_e\}.
\end{aligned}$$

and $\deg(x) = m$ for all $x \in \mathcal{B}_m$, $m = 0, 1, 2$.

If $e = 2$, then the homogeneous basis given by the proposition is simply $\{e(01), y_2 e(01)\}$.

Proof. Note that by Lemma 5.3 and Theorem 4.1 we have $R_{e-1}^{\Lambda_0} = (R_{e-1}^{\Lambda_0})_{\{0\}}$, so $y_1 = \dots = y_{e-1} = 0$ in $R_{e-1}^{\Lambda_0}$. Since the graded algebra homomorphism ι_{n-1}^n (see §3.1) is well defined, we have $y_1 = \dots = y_{e-1} = 0$ in $R_e^{\Lambda_0}$ as well. By Lemma 5.2 and Theorem 4.1, the following statements are true for any $\mathbf{i}, \mathbf{j} \in I^{\emptyset,1}$:

1. If $i_e = j_e$, then $\text{qdim}(e(\mathbf{j})R_{\delta}^{\Lambda_0}e(\mathbf{i})) = 1 + q^2$.
2. If $i_e = j_e \pm 1$, then $\text{qdim}(e(\mathbf{j})R_{\delta}^{\Lambda_0}e(\mathbf{i})) = q$.
3. If $j_e \notin \{i_e, i_e - 1, i_e + 1\}$, then $e(\mathbf{j})R_{\delta}^{\Lambda_0}e(\mathbf{i}) = 0$.

In particular, $R_{\delta}^{\Lambda_0} = (R_{\delta}^{\Lambda_0})_{\{0,1,2\}}$. By Theorem 3.5(i), it follows that, whenever $\mathbf{i}, \mathbf{j} \in I^{\emptyset,1}$ and $i_e = j_e$, the vector space $e(\mathbf{j})R_{\delta}^{\Lambda_0}e(\mathbf{i})$ is spanned by $\{\psi_{w_{j,i}}e(\mathbf{i}), \psi_{w_{j,i}}y_e e(\mathbf{i})\}$ and hence, by comparing graded dimensions, that these elements form a basis of $e(\mathbf{j})R_{\delta}^{\Lambda_0}e(\mathbf{i})$ and have degrees 0 and 2 respectively (the latter fact is also easily seen directly from definitions). Similarly, if $j_e = i_e \pm 1$ (and hence $e > 2$), then the singleton set $\{\psi_{w_{j,i}}e(\mathbf{i})\}$ spans the vector space $e(\mathbf{j})R_{\delta}^{\Lambda_0}e(\mathbf{i})$ and hence forms a basis of this space; moreover, the unique element of this set has degree 1. The proposition follows from the above assertions. \square

Lemma 5.5. *Assume that $e > 2$. Then*

- (i) *If $\mathbf{i}, \mathbf{j} \in I^{\emptyset,1}$ and $j_e \in \{i_e - 1, i_e + 1\}$, then*

$$\psi_{w_{i,j}}\psi_{w_{j,i}}e(\mathbf{i}) = \begin{cases} -y_e e(\mathbf{i}) & \text{if } j_e = i_e - 1, \\ y_e e(\mathbf{i}) & \text{if } j_e = i_e + 1. \end{cases}$$

- (ii) *The algebra $R_{\delta}^{\Lambda_0}$ is generated by $(R_{\delta}^{\Lambda_0})_{\{0,1\}}$.*

Proof. (i) It is easy to see that there exists $\mathbf{k} \in I^{\emptyset,1}$ such that $k_e = i_e$ and $k_{e-1} = j_e$. Then $\mathbf{k}' := s_{e-1}\mathbf{k}$ also belongs to $I^{\emptyset,1}$, and $k'_e = j_e$. We have $w_{i,j} = w_{i,k}s_{e-1}w_{\mathbf{k}',j}$ and, since $s_{e-1} \in {}^{(e-1,1)}\mathcal{D}_e^{(e-1,1)}$ and $w_{i,k}, w_{\mathbf{k}',j} \in \mathfrak{S}_{(e-1,1)}$, it follows that $\psi_{w_{i,j}}e_{\delta} = \psi_{w_{i,k}}\psi_{e-1}\psi_{w_{\mathbf{k}',j}}e_{\delta}$. Similarly, $\psi_{w_{j,i}}e_{\delta} = \psi_{w_{j,k'}}\psi_{e-1}\psi_{w_{\mathbf{k},i}}e_{\delta}$. Since $i_e = k_e$, we have $\deg(\psi_{w_{\mathbf{k},i}}e(\mathbf{i})) = 0$, and hence, applying repeatedly the case $i_r \neq i_{r+1}$ of the relation (3.11) and using the fact that $w_{i,k} = w_{\mathbf{k},i}^{-1}$, we obtain $\psi_{w_{i,k}}\psi_{w_{\mathbf{k},i}}e(\mathbf{i}) = e(\mathbf{i})$. Similarly, $\psi_{w_{j,k'}}\psi_{w_{\mathbf{k}',j}}e(\mathbf{k}') = e(\mathbf{k}')$. Set $\varepsilon = 1$ if $j_e = i_e + 1$ and $\varepsilon = -1$ if $j_e = i_e - 1$. Using the above equalities together with the fact that $y_{e-1} = 0$ (see the proof of Proposition 5.4) and that $\psi_{w_{\mathbf{k},i}}$ commutes with y_e (as $w_{\mathbf{k},i}(e) = e$), we obtain

$$\begin{aligned} \psi_{w_{i,j}}\psi_{w_{j,i}}e(\mathbf{i}) &= \psi_{w_{i,k}}\psi_{e-1}\psi_{w_{\mathbf{k}',j}}\psi_{w_{j,k'}}\psi_{e-1}\psi_{w_{\mathbf{k},i}}e(\mathbf{i}) \\ &= \psi_{w_{i,k}}\psi_{e-1}\psi_{w_{\mathbf{k}',j}}\psi_{w_{j,k'}}e(\mathbf{k}')\psi_{e-1}\psi_{w_{\mathbf{k},i}}e(\mathbf{i}) \\ &= \psi_{w_{i,k}}\psi_{e-1}^2e(\mathbf{k}')\psi_{w_{\mathbf{k},i}}e(\mathbf{i}) \\ &= \varepsilon\psi_{w_{i,k}}(y_e - y_{e-1})\psi_{w_{\mathbf{k},i}}e(\mathbf{i}) \\ &= \varepsilon\psi_{w_{i,k}}y_e\psi_{w_{\mathbf{k},i}}e(\mathbf{i}) = \varepsilon\psi_{w_{i,k}}\psi_{w_{\mathbf{k},i}}y_e e(\mathbf{i}) = \varepsilon y_e e(\mathbf{i}). \end{aligned}$$

- (ii) Since $e \geq 3$, for each $\mathbf{i} \in I^{\emptyset,1}$, there exists $\mathbf{j} \in I^{\emptyset,1}$ such that $j_e \in \{i_e - 1, i_e + 1\}$. The result now follows from (i) and Proposition 5.4. \square

6. WREATH PRODUCT RELATIONS IN A QUOTIENT OF A KLR ALGEBRA

In this section we construct a unital graded algebra homomorphism $\Theta: R_{\delta}^{\Lambda_0} \wr \mathfrak{S}_d \rightarrow e_{\delta d} \overline{R}_{d\delta} e_{\delta d}$ (cf. (3.26)). As is mentioned in §3.3, we adapt ideas from [17, Section 1] in order to define the images τ_r of elementary transpositions $s_r \in \mathfrak{S}_d$ under Θ . The elements τ_r are defined by Equation (6.9), and their needed properties are summarised in Theorem 6.14. The present set-up is quite different from that of [17]: in particular, the “error terms”

ϵ_r appearing in (6.9) have no analogue in [17]. Consequently, the presentation below is largely self-contained.

6.1. The intertwiners φ_w . Fix $n \geq 0$. We recall necessary facts from [17, §1.3.1]. For $1 \leq r < n$, define $\varphi_r \in R_n$ by

$$(6.1) \quad \varphi_r e(\mathbf{i}) = \begin{cases} (\psi_r y_r - y_r \psi_r) e(\mathbf{i}) = (\psi_r (y_r - y_{r+1}) + 1) e(\mathbf{i}) & \text{if } i_r = i_{r+1}, \\ \psi_r e(\mathbf{i}) & \text{if } i_r \neq i_{r+1} \end{cases}$$

for all $\mathbf{i} \in I^n$.

If $w = s_{r_1} \cdots s_{r_m}$ is a reduced expression in \mathfrak{S}_n , define $\varphi_w := \varphi_{r_1} \cdots \varphi_{r_m}$. It follows from part (ii) of the following lemma and Matsumoto's theorem that φ_w depends only on w , not on the choice of the reduced expression. In particular, we note that

$$(6.2) \quad \varphi_v \varphi_w = \varphi_{vw}$$

whenever $v, w \in \mathfrak{S}_n$ and $\ell(vw) = \ell(v) + \ell(w)$. Also, we will repeatedly use the fact that $\varphi_w e(\mathbf{i}) = e(w\mathbf{i}) \varphi_w$ for all $\mathbf{i} \in I^n$, $w \in \mathfrak{S}_n$.

Lemma 6.1. [17, Lemma 1.3.1] *For $1 \leq r < n$, $1 \leq t \leq n$, $w \in \mathfrak{S}_n$ and $\mathbf{i} \in I^n$,*

- (i) $\varphi_r^2 e(\mathbf{i}) = (L_{i_r, i_{r+1}}(y_r, y_{r+1}) + \delta_{i_r, i_{r+1}}) e(\mathbf{i})$;
- (ii) $\varphi_r \varphi_{r+1} \varphi_r = \varphi_{r+1} \varphi_r \varphi_{r+1}$ if $r < n-1$;
- (iii) $\varphi_w y_t = y_{w(t)} \varphi_w$;
- (iv) if $1 \leq k < n$ and $w(k+1) = w(k) + 1$, then $\varphi_w \psi_k = \psi_{w(k)} \varphi_w$;
- (v) $\varphi_{w^{-1}} \varphi_w e(\mathbf{i}) = \prod_{\substack{1 \leq a < b \leq n \\ w(a) > w(b)}} (L_{i_a, i_b}(y_a, y_b) + \delta_{i_a, i_b}) e(\mathbf{i})$.

Suppose now that $n = 2e$. Recalling the definition before Corollary 3.9, consider the subalgebra R'_{δ^2} of $R_{2\delta}$.

Lemma 6.2. *Let $w = (1, e+1)(2, e+2) \cdots (e, 2e) \in \mathfrak{S}_{2e}$, and let K be the ideal of $F[y_1, \dots, y_{2e}]$ generated by the set $\{y_r - y_t \mid 1 \leq r, t \leq e\} \cup \{y_r - y_t \mid e+1 \leq r, t \leq 2e\}$. Then, in $R_{2\delta}$, we have*

$$(6.3) \quad \varphi_w e_{\delta, \delta} \in \psi_w((y_1 - y_{e+1})^e + K) e_{\delta, \delta} + \sum_{v \in \mathcal{D}_{2e}^{(e, e)} \setminus \{w\}} \psi_v R'_{\delta, \delta}.$$

Proof. The idea of the proof is the same as that of [17, Proposition 1.4.4]. Note that w is fully commutative (see [20, Lemma 3.17]), and hence ψ_w does not depend on the choice of a reduced expression for w . Let $\mathbf{j}, \mathbf{k} \in I^\delta$ and $\mathbf{i} = \mathbf{j}\mathbf{k}$. It is enough to show that

$$\varphi_w e(\mathbf{i}) \in \psi_w((y_1 - y_{e+1})^e + K) e(\mathbf{i}) + \sum_{v \in \mathcal{D}_{2e}^{(e^2)} \setminus \{w\}} \psi_v R'_{\delta, \delta}$$

for all such \mathbf{i} . Let $w = s_{r_{e^2}} \cdots s_{r_2} s_{r_1}$ be a reduced expression, and let $u_k = s_{r_{k-1}} \cdots s_{r_1}$ for $1 \leq k \leq e^2$. We have

$$(6.4) \quad \varphi_w e(\mathbf{i}) = \varphi_{r_{e^2}} \cdots \varphi_{r_1} e(\mathbf{i}).$$

For each k , one can replace the multiple φ_{r_k} by $\varphi_{r_k} e(u_k \mathbf{i})$ without changing the value of the expression on the right-hand side.

Let k run in the decreasing order through the elements of $\{1, \dots, e^2\}$ satisfying $(u_k \mathbf{i})_{r_k} = (u_k \mathbf{i})_{r_k+1}$. Note that there are exactly e such values of k since each of \mathbf{j} and \mathbf{k} is a permutation of $(0, 1, \dots, e-1)$. For each such k , we have $\varphi_{r_k} e(u_k \mathbf{i}) = \psi_{r_k} (y_{r_k} - y_{r_k+1}) e(u_k \mathbf{i}) + e(u_k \mathbf{i})$. Consequently, the product (6.4) decomposes as a sum of two summands, which correspond to $\psi_{r_k} (y_{r_k} - y_{r_k+1}) e(u_k \mathbf{i})$ and $e(u_k \mathbf{i})$ respectively. By Lemma 6.1(iii), the first summand does not change if we remove the factor $(y_{r_k} - y_{r_k+1})$ and instead insert $y_{u_k^{-1}(r_k)} - y_{u_k^{-1}(r_k+1)}$ at the right end of the product (note that $u_k^{-1}(r_k) \leq e$ and

$u_k^{-1}(r_k + 1) > e$). After these manipulations are performed for all such k in the decreasing order, we have decomposed the product (6.4) into 2^e summands. Of these, all the summands for which the second option was chosen at least once belong to $\sum_{v \in \mathcal{D}_{2^e}^{(e^2)} \setminus \{w\}} \psi_v R_{\delta, \delta}$ by Theorem 3.5(ii) together with the argument used to prove Corollary 3.6. The remaining summand belongs to $\psi_w((y_1 - y_{e+1})^e e(\mathbf{i}) + K)$, as in all cases we have $y_{u_k^{-1}(r_k)} - y_{u_k^{-1}(r_{k+1})} \in y_1 - y_{e+1} + K$. Thus, $\varphi_w e(\mathbf{i}) \in \psi_w((y_1 - y_{e+1})^e + K) e_{\delta, \delta} + a$ for some $a = \sum_{v \in \mathcal{D}_{2^e}^{(e^2)} \setminus \{w\}} \psi_v a_v$, with $a_v \in R_{\delta, \delta} e(\mathbf{i})$ for each v . Since $\varphi_w e(\mathbf{i}) \in R'_{2\delta}$ and $\psi_w((y_1 - y_{e+1})^e + K) e(\mathbf{i}) \subset R'_{2\delta}$, we have $a \in R'_{2\delta}$. By Theorem 3.5(i), we can write $a_v = \sum_{z \in \mathfrak{S}_{(e, e)}} \psi_z e(\mathbf{i}) x_{v, z}$ with $x_{v, z} \in F[y_1, \dots, y_{2e}]$, so that $a = \sum_{v \in \mathcal{D}_{2^e}^{(e^2)} \setminus \{w\}} \sum_{z \in \mathfrak{S}_{(e, e)}} \psi_v \psi_z e(\mathbf{i}) x_{v, z}$. We may assume that for any such v and z the reduced expression for the definition of ψ_{vz} is chosen in such a way that $\psi_{vz} = \psi_v \psi_z$. Then, by Theorem 3.5(i) and Proposition 3.8, $x_{v, z} \in R'_{\delta, \delta}$ for all v and z , and hence $a \in \sum_{v \in \mathcal{D}_{2^e}^{(e^2)} \setminus \{w\}} \psi_v R'_{\delta, \delta}$. \square

6.2. Quotients of R_δ . Throughout this subsection, we view ψ_r and y_t for $1 \leq r < e$ and $1 \leq t \leq e$ as elements of either R_δ or $R_\delta^{\Lambda_0}$ (depending on the context) via the natural projections $R_n \twoheadrightarrow R_\delta$ and $R_n^{\Lambda_0} \twoheadrightarrow R_\delta^{\Lambda_0}$. Let V be the 2-sided ideal of R_δ generated by $\mathcal{E}_1 = \{e(\mathbf{i}) \mid \mathbf{i} \in I^\delta \setminus I^{\emptyset, 1}\}$, so that $\overline{R}_\delta = R_\delta/V$ (cf. §3.3). Let $\overline{\cdot} : R_\delta \twoheadrightarrow \overline{R}_\delta$ be the natural projection.

Lemma 6.3. *We have $\overline{\psi}_1 = 0$.*

Proof. If not, then by the relation (3.5) there exists $\mathbf{i} \in I^{\emptyset, 1}$ such that $s_1 \mathbf{i} \in I^{\emptyset, 1}$. Since every $\mathbf{j} \in I^{\emptyset, 1}$ satisfies $j_1 = 0$ and $j_2 \neq 0$, this is impossible. \square

Lemma 6.4. *There is a unital graded algebra homomorphism $\eta : R_\delta^{\Lambda_0} \rightarrow \overline{R}_\delta$ given by*

$$\eta(e(\mathbf{i})) = \overline{e(\mathbf{i})}, \quad \eta(\psi_r) = \overline{\psi}_r, \quad \eta(y_t) = \overline{y}_t - \overline{y}_1$$

for $\mathbf{i} \in I^\delta$, $1 \leq r < e$ and $1 \leq t \leq e$.

Proof. To begin with, η is a homomorphism from the free algebra on the standard generators of $R_\delta^{\Lambda_0}$ to \overline{R}_δ . It is immediate that η respects the defining relations of $R_\delta^{\Lambda_0}$ (including the cyclotomic relation $y_1^{\delta_1, 0} e(\mathbf{i}) = 0$), with the possible exception of relations (3.8)–(3.10) (note that $\eta(y_{r+1} - y_r) = \overline{y}_{r+1} - \overline{y}_r$ for $1 \leq r < e$ and that $\eta(y_1) = 0$).

Recall that $i_r \neq i_{r+1}$ for all $\mathbf{i} \in I^\delta$ and $1 \leq r < d$. We have $\eta(\psi_1) = 0$ by Lemma 6.4, so both sides of (3.8), (3.9), (3.10) are mapped by η to 0 for $r = 1$. For $r > 1$, we have

$$\eta(y_{r+1} \psi_r) = (\overline{y}_{r+1} - \overline{y}_1) \overline{\psi}_r = (\overline{y}_{r+1} \overline{\psi}_r - \overline{\psi}_r \overline{y}_1) = \overline{\psi}_r (\overline{y}_r - \overline{y}_1) = \eta(\psi_r y_r),$$

where the second equality is due to (3.8) and the third one is due to (3.9). Relations (3.8) and (3.9) for $r > 1$ are checked similarly. \square

Lemma 6.5. *Let z be an indeterminate, and view $F[z]$ as a graded algebra with $\deg(z) = 2$. We have a graded algebra isomorphism $R_\delta^{\Lambda_0} \otimes F[z] \xrightarrow{\sim} \overline{R}_\delta$ given by $a \otimes z^m \mapsto \eta(a) \overline{y}_1^m$ for all $m \geq 0$ and $a \in R_\delta^{\Lambda_0}$.*

Proof. By Lemma 6.3 and the defining relations of \overline{R}_δ , the element \overline{y}_1 centralises $\eta(R_\delta^{\Lambda_0})$. Hence, it follows from Lemma 6.4 that we have a graded unital algebra homomorphism $\xi : R_\delta^{\Lambda_0} \otimes F[z] \rightarrow \overline{R}_\delta$ defined as in the statement of the lemma.

By an observation in [17, §1.3.2] (cf. (3.23)), there is a unital algebra homomorphism $R_\delta \rightarrow R_\delta^{\Lambda_0} \otimes F[z]$ given by

$$(6.5) \quad e(\mathbf{i}) \mapsto e(\mathbf{i}) \otimes 1, \quad \psi_r \mapsto \psi_r \otimes 1, \quad y_t \mapsto y_t \otimes 1 + 1 \otimes z$$

for $\mathbf{i} \in I^\delta$, $1 \leq r < e$, $1 \leq t \leq e$. By Theorem 4.1, we have $e(\mathbf{i}) = 0$ in $R_\delta^{\Lambda_0}$ for $\mathbf{i} \in I^\delta \setminus I^{\emptyset, 1}$, so the map (6.5) factors through \overline{R}_δ . It is easy to check that the resulting homomorphism $\overline{R}_\delta \rightarrow R_\delta^{\Lambda_0} \otimes F[z]$ is both a left and a right inverse to ξ . \square

Using the fact that \overline{R}_δ is nonnegatively graded and Proposition 5.4, we immediately deduce the following result.

Corollary 6.6. *The map η restricts to a vector space isomorphism from $(R_\delta^{\Lambda_0})_{\{0,1\}}$ onto $(\overline{R}_\delta)_{\{0,1\}}$. Moreover, $(\overline{R}_\delta)_{\{0,1\}} = \sum_{\mathbf{i} \in I^{\emptyset,1}} \sum_{v \in \mathfrak{S}_e} F\psi_v \overline{e(\mathbf{i})}$.*

Let K be the left ideal of \overline{R}_δ generated by the set $\{\overline{y}_k - \overline{y}_t \mid 1 \leq k, t \leq e\}$. For $\mathbf{i} \in I^\delta$, $1 \leq r < e$ and $1 \leq r, t \leq e$, we have $(\overline{y}_k - \overline{y}_t) \overline{\psi_r e(\mathbf{i})} = \overline{\psi_r e(\mathbf{i})} (\overline{y}_{s_r(k)} - \overline{y}_{s_r(t)})$. Hence, K is a 2-sided ideal of \overline{R}_δ .

Lemma 6.7. *Let z be an indeterminate. The F -linear map $(\overline{R}_\delta)_{\{0,1\}} \otimes F[z] \rightarrow \overline{R}_\delta/K$ given by $a \otimes z^m \mapsto a \overline{y}_1^m + K$ is an isomorphism of vector spaces.*

Proof. Using Proposition 5.4, we see that the image of K in $R_\delta^{\Lambda_0} \otimes F[z]$ under the inverse of the isomorphism of Lemma 6.5 is equal to $(R_\delta^{\Lambda_0})_{\{2\}} \otimes F[z]$. Hence, the map $(R_\delta^{\Lambda_0})_{\{0,1\}} \otimes F[z] \rightarrow \overline{R}_\delta/K$ given by $a \otimes z^m \mapsto \eta(a) \overline{y}_1^m + K$ is an isomorphism of vector spaces. Due to Corollary 6.6, the result follows. \square

6.3. A homomorphism from $R_\delta^{\Lambda_0} \wr \mathfrak{S}_d$ to $e_{\delta d} \overline{R}_{d\delta} e_{\delta d}$. Fix an arbitrary integer $d \geq 0$. Let \mathcal{V} be the 2-sided ideal of $R_{\delta d}$ generated by the set

$$\{e(\mathbf{i}^{(1)} \dots \mathbf{i}^{(d)}) \mid (\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(d)}) \in (I^\delta)^{\times d} \setminus (I^{\emptyset,1})^{\times d}\}.$$

Let \mathcal{U} be the 2-sided ideal of $R_{d\delta}$ generated by $\{e(\mathbf{i}) \mid \mathbf{i} \in I^{d\delta} \setminus \mathcal{E}_d\}$ (cf. (3.24)), so that $\overline{R}_{d\delta} = R_{d\delta}/\mathcal{U}$. Define $\overline{R}_{\delta d}$ to be the image of $R_{\delta d}$ under the natural projection $R_{d\delta} \twoheadrightarrow \overline{R}_{d\delta}$.

Lemma 6.8. *We have:*

- (i) $\mathcal{U} e_{\delta d} = R_{d\delta} \mathcal{V}$;
- (ii) $\mathcal{U} \cap R_{\delta d} = \mathcal{V}$;
- (iii) $\overline{R}_{d\delta} e_{\delta d}$ is a free right $\overline{R}_{\delta d}$ -module with basis $\{\overline{\psi}_w \mid w \in \mathcal{D}_{de}^{(e^d)}\}$.

Proof. (i) By Lemma 4.5, $\mathcal{V} \subset \mathcal{U}$. Hence, $R_{d\delta} \mathcal{V} \subset \mathcal{U} e_{\delta d}$, as $\mathcal{V} e_{\delta d} = \mathcal{V}$. By Corollary 3.6, $R_{d\delta} e_{\delta d} = \sum_{w \in \mathcal{D}_{de}^{(e^d)}} \psi_w R_{\delta d}$. Since $R_{\delta d} = \sum_{\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(d)} \in I^{\emptyset,1}} e(\mathbf{i}^{(1)} \dots \mathbf{i}^{(d)}) R_{\delta d} + \mathcal{V}$, we have $R_{d\delta} e_{\delta d} = \sum_{\mathbf{i} \in \mathcal{E}_d} e(\mathbf{i}) R_{d\delta} e_{\delta d} + R_{d\delta} \mathcal{V}$. Hence, for any $\mathbf{i} \in I^{d\delta} \setminus \mathcal{E}_d$, we have $e(\mathbf{i}) R_{d\delta} e_{\delta d} \subset R_{d\delta} \mathcal{V}$. Since $\mathcal{U} = \sum_{\mathbf{i} \in I^{d\delta} \setminus \mathcal{E}_d} R_{d\delta} e(\mathbf{i}) R_{d\delta}$, the inclusion $\mathcal{U} e_{\delta d} \subset R_{d\delta} \mathcal{V}$ follows.

(ii) and (iii) follow from (i) and Corollary 3.6. \square

Due to Lemma 6.8(ii), $\overline{R}_{\delta d}$ is naturally identified with $R_{\delta d}/\mathcal{V}$. Hence, the isomorphism $\iota_{\delta d}: R_\delta^{\otimes d} \xrightarrow{\sim} R_{\delta d}$ induces an algebra isomorphism $\iota: \overline{R}_\delta^{\otimes d} \xrightarrow{\sim} \overline{R}_{\delta d}$.

Throughout the rest of the section, symbols of the form $\psi_r, \psi_w, \varphi_r, \varphi_w, e(\mathbf{i}), e_\alpha$ and y_t that would previously be interpreted as elements R_{de} represent their images in $\overline{R}_{d\delta}$. It follows from Lemma 4.5 that, if $\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(d)} \in I^e$, then we have $e(\mathbf{i}^{(1)} \dots \mathbf{i}^{(d)}) = 0$ in $\overline{R}_{d\delta}$ unless $\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(d)} \in I^{\emptyset,1}$. Hence,

$$(6.6) \quad e_{\delta d} = \sum_{\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(d)} \in I^{\emptyset,1}} e(\mathbf{i}^{(1)} \dots \mathbf{i}^{(d)}).$$

For $1 \leq r < d$, let

$$(6.7) \quad w_r = ((r-1)e+1, re+1)((r-1)e+2, re+2) \cdots (re, (r+1)e) \in \mathfrak{S}_{de}.$$

This element is fully commutative (cf. the proof of Lemma 6.2), so $\sigma_r := \psi_{w_r} \in \overline{R}_{d\delta}$ is well defined in the sense that it does not depend on the choice of a reduced expression for w_r . Let B_d be the subgroup of \mathfrak{S}_{de} generated by the elements w_r , $1 \leq r < d$. Then we have a group isomorphism $o: \mathfrak{S}_d \rightarrow B_d$, $s_r \mapsto w_r$, and $B_d \subset {}^{(e^d)}\mathcal{D}_{ed}^{(e^d)}$ (cf. [21, §4.1]).

For each $u \in \mathfrak{S}_d$, choose a reduced expression $u = s_{r_1} \cdots s_{r_m}$ and define $\sigma_u = \sigma_{r_1} \cdots \sigma_{r_m}$. We may assume that $\sigma_u = \psi_{o(u)}$, as the decomposition $o(u) = w_{r_1} \cdots w_{r_m}$ can be refined

to a reduced expression for $o(u)$ in \mathfrak{S}_{de} . Note that $\sigma_u e_{\delta^d} = e_{\delta^d} \sigma_u$ for all $u \in \mathfrak{S}_d$. We have $\{1, \dots, de\} = \bigsqcup_{r=1}^d X_r$, where $X_r := \{(r-1)e + 1, \dots, re\}$.

Lemma 6.9. (i) We have $e_{\delta^d} \bar{R}_{d\delta} e_{\delta^d} = \sum_{u \in \mathfrak{S}_d} \sigma_u \bar{R}_{\delta^d}$.

(ii) For all $u \in \mathfrak{S}_d$, we have $\sigma_u e_{\delta^d} \in (\bar{R}_{d\delta})_{\{0\}}$.

(iii) The algebra $e_{\delta^d} \bar{R}_{d\delta} e_{\delta^d}$ is nonnegatively graded.

Proof. For (ii), it is enough to consider the case when $u = s_r$ for some $r \in \{1, \dots, d-1\}$ because each σ_r centralises e_{δ^d} . The fact that $\sigma_r e_{\delta^d}$ is homogeneous of degree 0 is easily verified using (6.6). Part (iii) follows from (i) and (ii) because $\bar{R}_{\delta^d} = \iota(\bar{R}_{\delta}^{\otimes d})$ is nonnegatively graded.

Thus, it remains only to prove (i). By Theorem 3.5(i), we have $\bar{R}_{d\delta} e_{\delta^d} = \sum_{w \in \mathcal{D}_{de}^{(e^d)}} \psi_w \bar{R}_{\delta^d}$.

If $w \in B_d$, say, $w = o(u)$, then $e_{\delta^d} \psi_w \bar{R}_{\delta^d} = e_{\delta^d} \sigma_u \bar{R}_{\delta^d} = \sigma_u \bar{R}_{\delta^d}$. So it will suffice to prove the following claim: if $w \in \mathcal{D}_{de}^{(e^d)} \setminus B_d$, then $e_{\delta^d} \psi_w \bar{R}_{\delta^d} = 0$.

If not, then by (6.6) there exist $\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(d)} \in I^{\emptyset, 1}$ such that $w(\mathbf{i}^{(1)} \dots \mathbf{i}^{(d)}) = \mathbf{j}^{(1)} \dots \mathbf{j}^{(d)}$ for some $\mathbf{j}^{(1)}, \dots, \mathbf{j}^{(d)} \in I^{\emptyset, 1}$. Since $w \in \mathcal{D}_{de}^{(e^d)} \setminus B_d$, there exist $k \in \{1, \dots, d\}$ and $t \in \{2, \dots, e\}$ such that $w((k-1)e + 1) \in X_l$ and $w((k-1)e + t) \in X_m$ for some $m > l$. Let such k, t be chosen so that m is greatest possible. Let $r \in \{1, \dots, d\}$ be such that $w((r-1)e + 1) = (m-1)e + 1$ (note that $w^{-1}((m-1)e + 1) \equiv 1 \pmod{e}$ because the only zeros in $\mathbf{i}^{(1)} \dots \mathbf{i}^{(d)}$ occur in positions with numbers congruent to 1 modulo e). Note that $r \neq k$, as $(k-1)e + 1$ and $(r-1)e + 1$ have distinct images under w . By maximality of m , we have $w(X_r) \subset X_m$, whence $w(X_r) = X_m$. This contradicts the fact that $w((k-1)e + t) \in X_m$. \square

Recall the ideal K of \bar{R}_{δ} defined in §6.2 and let

$$\mathcal{K} = \sum_{r=1}^d \iota(\bar{R}_{\delta}^{\otimes r-1} \otimes K \otimes \bar{R}_{\delta}^{\otimes d-r}) \subset \bar{R}_{\delta^d}.$$

In other words, \mathcal{K} is the 2-sided (or, equivalently, left) ideal of \bar{R}_{δ^d} generated by the elements of the form $(y_r - y_t)e_{\delta^d}$ with $r, t \in X_k$ for $1 \leq k \leq d$. We have a natural isomorphism

$$(6.8) \quad (\bar{R}_{\delta}/K)^{\otimes d} \xrightarrow{\sim} \bar{R}_{\delta^d}/\mathcal{K}, \quad (x_1 + K) \otimes \dots \otimes (x_d + K) \mapsto \iota(x_1 \otimes \dots \otimes x_d) + \mathcal{K}$$

for $x_1, \dots, x_d \in \bar{R}_{\delta}$.

Let $\mathcal{J} = e_{\delta^d} \bar{R}_{d\delta} \mathcal{K}$, so that \mathcal{J} is a left ideal of $e_{\delta^d} \bar{R}_{d\delta} e_{\delta^d}$. The map $\bar{R}_{\delta^d} \rightarrow e_{\delta^d} \bar{R}_{d\delta} e_{\delta^d} / \mathcal{J}$ given by $a \mapsto a + \mathcal{J}$ has kernel \mathcal{K} and image $(\bar{R}_{\delta^d} + \mathcal{J})/\mathcal{J}$. For any $a + \mathcal{J} \in e_{\delta^d} \bar{R}_{d\delta} e_{\delta^d} / \mathcal{J}$ and $b \in \bar{R}_{\delta^d}$, the product $(a + \mathcal{J})(b + \mathcal{J}) = ab + \mathcal{J}$ is well-defined. Thus, $e_{\delta^d} \bar{R}_{d\delta} e_{\delta^d} / \mathcal{J}$ is naturally an $(e_{\delta^d} \bar{R}_{d\delta} e_{\delta^d}, (\bar{R}_{\delta^d} + \mathcal{J})/\mathcal{J})$ -bimodule. It follows from Lemmas 6.8(iii) and 6.9(i) that $e_{\delta^d} \bar{R}_{d\delta} e_{\delta^d} / \mathcal{J}$ is free as a right $(\bar{R}_{\delta^d} + \mathcal{J})/\mathcal{J}$ -module with basis $\{\sigma_u + \mathcal{J} \mid u \in \mathfrak{S}_d\}$. In particular, $(\bar{R}_{\delta^d} + \mathcal{J})/\mathcal{J}$ is a free right $(\bar{R}_{\delta^d} + \mathcal{J})/\mathcal{J}$ -submodule of $e_{\delta^d} \bar{R}_{d\delta} e_{\delta^d} / \mathcal{J}$. The subspace $(\bar{R}_{\delta^d} + \mathcal{J})/\mathcal{J}$ also has an F -algebra structure arising from the isomorphism $\bar{R}_{\delta^d}/\mathcal{K} \xrightarrow{\sim} (\bar{R}_{\delta^d} + \mathcal{J})/\mathcal{J}$, $b + \mathcal{K} \mapsto b + \mathcal{J}$, and the right $(\bar{R}_{\delta^d} + \mathcal{J})/\mathcal{J}$ -module structure on $(\bar{R}_{\delta^d} + \mathcal{J})/\mathcal{J}$ induced by this algebra structure coincides with the aforementioned module structure. These facts are used repeatedly in the sequel.

Define

$$S = ((\bar{R}_{\delta})_{\{0,1\}})^{\otimes d} \subset \bar{R}_{\delta}^{\otimes d} \quad \text{and} \\ T = \sum_{u \in \mathfrak{S}_d} \sigma_u \iota(S) \subset e_{\delta^d} \bar{R}_{d\delta} e_{\delta^d}.$$

Note that T is a graded vector subspace of $\overline{R}_{d\delta}$. For all $1 \leq r \leq d$, set

$$z_r = y_{(r-1)e+1}e_{\delta^d} + \mathcal{J} \in (\overline{R}_{\delta^d} + \mathcal{J})/\mathcal{J}$$

These elements commute and generate the subalgebra $F[z_1, \dots, z_d]$ of $(\overline{R}_{\delta^d} + \mathcal{J})/\mathcal{J}$.

Lemma 6.10. *Let $\tilde{z}_1, \dots, \tilde{z}_d$ be algebraically independent indeterminates. Consider the F -algebra $F[\tilde{z}_1, \dots, \tilde{z}_d]$, graded by the rule $\deg(\tilde{z}_r) = 2$ for $1 \leq r \leq d$.*

- (i) *The F -linear map $F\mathfrak{S}_d \otimes S \otimes F[\tilde{z}_1, \dots, \tilde{z}_d] \rightarrow e_{\delta^d}\overline{R}_{d\delta}e_{\delta^d}/\mathcal{J}$ given by $u \otimes a \otimes g \mapsto \sigma_u \iota(a)g(z_1, \dots, z_d)$ (for $u \in \mathfrak{S}_d$) is an isomorphism of vector spaces.*
- (ii) *The F -linear map $T \otimes F[\tilde{z}_1, \dots, \tilde{z}_d] \rightarrow e_{\delta^d}\overline{R}_{d\delta}e_{\delta^d}/\mathcal{J}$ given by $t \otimes g \mapsto tg(z_1, \dots, z_d)$ is a graded vector space isomorphism.*

Proof. Part (i) is established by combining Lemma 6.7, the isomorphism (6.8) and the fact that $\{\sigma_u + \mathcal{J} \mid u \in \mathfrak{S}_d\}$ is a free generating set of $e_{\delta^d}\overline{R}_{d\delta}e_{\delta^d}/\mathcal{J}$ as a right $(\overline{R}_{\delta^d} + \mathcal{J})/\mathcal{J}$ -module. The map in part (ii) is easily seen to be a graded homomorphism. The fact that it is bijective clearly follows from (i). \square

Corollary 6.11. *We have $(e_{\delta^d}\overline{R}_{d\delta}e_{\delta^d})_{\{0,1\}} \subset T$.*

Proof. This follows from Lemmas 6.9(iii) and 6.10(ii). \square

If $\mu = (\mu_1, \dots, \mu_l)$ is a composition of $d\delta$, let $\bar{\iota}_\mu: R_{\mu_1} \otimes \dots \otimes R_{\mu_l} \rightarrow \overline{R}_{d\delta}$ be the composition of ι_μ with the natural projection $R_{d\delta} \twoheadrightarrow \overline{R}_{d\delta}$. Fix an integer r such that $1 \leq r < d$. Let $S^{(r)} = \bar{e}_\delta^{\otimes r-1} \otimes (\overline{R}_\delta)_{\{0,1\}} \otimes (\overline{R}_\delta)_{\{0,1\}} \otimes \bar{e}_\delta^{\otimes d-r-1} \subset S$.

Lemma 6.12. *We have $\varphi_{w_r}e_{\delta^d} + \mathcal{J} \in \sigma_r(z_r - z_{r+1})^e + \iota(S^{(r)})F[z_r - z_{r+1}]$.*

Proof. Let $\beta: R_{2\delta} \rightarrow \overline{R}_{d\delta}$ be the algebra homomorphism defined by

$$\beta(x) = \bar{\iota}_{\delta^{r-1}, 2\delta, \delta^{d-r-1}}(e_\delta^{\otimes r-1} \otimes x \otimes e_\delta^{\otimes d-r-1}).$$

We use Lemma 6.2 and apply β to both sides of (6.3). It follows from (6.6) and the relation (3.5) that $e_{\delta^d}\beta(\psi_v)e_{\delta^d} = 0$ in $\overline{R}_{d\delta}$ for all $v \in \mathcal{D}_{2e}^{(e^2)} \setminus \{1, w_1\}$, as the set $\{1, e+1\}$ is not v -invariant for any such v and every element of $I^{\emptyset,1}$ has 0 as its first entry. Since $e_{\delta^d}\varphi_{w_r}e_{\delta^d} = \varphi_{w_r}e_{\delta^d}$, we conclude that

$$\varphi_{w_r}e_{\delta^d} + \mathcal{J} \in \psi_{w_r}(z_r - z_{r+1})^e + Y,$$

in $e_{\delta^d}\overline{R}_{d\delta}e_{\delta^d}/\mathcal{J}$ where $Y = (\beta(R'_{\delta^2}) + \mathcal{J})/\mathcal{J}$. It follows from the definition of \mathcal{J} (and of \mathcal{K}) together with Corollary 3.9 applied to R'_{δ^2} that

$$Y \subset \sum_{v \in \mathfrak{S}_{(e,e)}} \beta(\psi_v e_{\delta^2})F[z_r - z_{r+1}].$$

By Corollary 6.6, we have $\beta(\psi_v e_{\delta^2}) + \mathcal{J} \in \beta(\iota_{\delta^2}((R_\delta)_{\{0,1\}} \otimes (R_\delta)_{\{0,1\}})) + \mathcal{J}$ for all $v \in \mathfrak{S}_{(e,e)}$, and the result follows. \square

Observing that $\deg(\varphi_{w_r}e_{\delta^d}) = 2e$, $\deg(\sigma_r e_{\delta^d}) = 0$, $S^{(r)} = (S^{(r)})_{\{0,1,2\}}$ and $\deg(z_r - z_{r+1}) = 2$, we deduce from Lemma 6.12 that

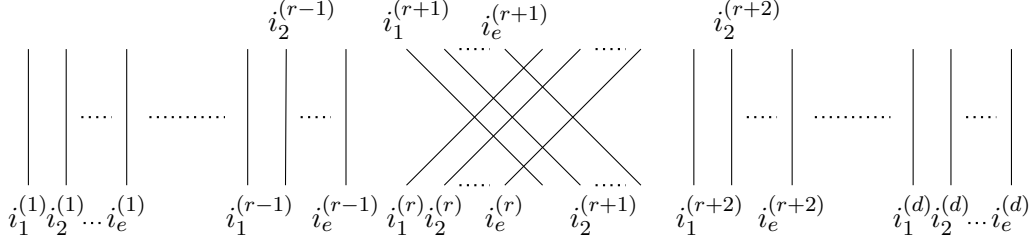
$$(6.9) \quad \varphi_{w_r}e_{\delta^d} + \mathcal{J} = \tau_r(z_r - z_{r+1})^e + \epsilon_r(z_r - z_{r+1})^{e-1}$$

for some elements $\tau_r \in \sigma_r e_{\delta^d} + \iota(S_{\{0\}}^{(r)}) \subset e_{\delta^d}\overline{R}_{d\delta}e_{\delta^d}$ and $\epsilon_r \in \iota(S_{\{2\}}^{(r)}) \subset \overline{R}_{\delta^d}$, which are determined uniquely due to Lemma 6.10(ii). By considering degrees, we see that $\epsilon_r \in \iota(\bar{e}_\delta^{\otimes r-1} \otimes (\overline{R}_\delta)_{\{1\}} \otimes (\overline{R}_\delta)_{\{1\}} \otimes \bar{e}_\delta^{\otimes d-r-1})$.

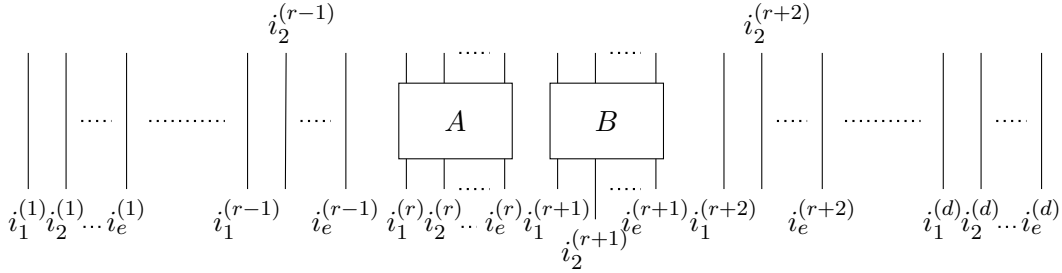
For $1 \leq t \leq d$, define a graded algebra homomorphism $\eta_t: R_\delta^{\Lambda_0} \rightarrow \overline{R}_{\delta^d}$ by $\eta_t(x) = \iota(\bar{e}_\delta^{\otimes t-1} \otimes \eta(x) \otimes \bar{e}_\delta^{\otimes d-t})$, where $\eta: R_\delta^{\Lambda_0} \rightarrow \overline{R}_\delta$ is given by Lemma 6.4. Note that, by Corollary 6.6, we have $(\overline{R}_\delta)_{\{1\}} = \eta((R_\delta^{\Lambda_0})_{\{1\}})$, and hence

$$(6.10) \quad \epsilon_r \in \eta_r((R_\delta^{\Lambda_0})_{\{1\}})\eta_{r+1}((R_\delta^{\Lambda_0})_{\{1\}}).$$

Remark 6.13. For any $\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(d)} \in I^\delta$, the element $\sigma_r e(\mathbf{i}^{(1)} \dots \mathbf{i}^{(d)})$ is the image of the element of $R_{d\delta}$ represented by the following Khovanov–Lauda diagram:



(cf. Remark 3.1). Any element of $S^{(r)}$ (including ϵ_r) is a linear combination of the images of diagrams of the form



where $\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(d)} \in I^\delta$ and A, B are diagrams representing elements of R_δ .

Now we can state the main result of this section. The equalities in parts (i)–(iv) of the following theorem are between elements of $\bar{R}_{d\delta}$.

Theorem 6.14. For $1 \leq r < d$, we have:

- (i) $\tau_r^2 = e_{\delta d}$;
- (ii) $\tau_r \tau_{r+1} \tau_r = \tau_{r+1} \tau_r \tau_{r+1}$ if $r < d - 1$;
- (iii) $\tau_r \tau_t = \tau_t \tau_r$ if $1 \leq t < d$ and $|t - r| \geq 2$.
- (iv) if $x \in R_\delta^{\Lambda_0}$, then $\eta_r(x) \tau_r = \tau_r \eta_{r+1}(x)$ and $\eta_{r+1}(x) \tau_r = \tau_r \eta_r(x)$.
- (v) $y_{re+1} e_{\delta d} \in \tau_r y_{(r-1)e+1} \tau_r + \mathcal{T}$, where \mathcal{T} is the unital subalgebra of $e_{\delta d} \bar{R}_{\delta d} e_{\delta d}$ generated by $T \cup \mathcal{K}$.

Remark 6.15. If one of parts (i),(ii),(v) of Theorem 6.14 holds for $d = 2$, then it holds in general, and if part (iv) holds for $d = 3$, then it holds in general. This may be seen by applying the graded algebra homomorphism $\zeta_{r,l}: \bar{R}_{l\delta} \rightarrow \bar{R}_{d\delta}$ induced by the map $R_{l\delta} \rightarrow \bar{R}_{d\delta}$, $x \mapsto \bar{\iota}_{\delta^{r-1}, l\delta, \delta^{d-r-l+1}}(e_\delta^{\otimes r-1} \otimes x \otimes e_\delta^{\otimes d-r-l+1})$, with $l = 3$ for part (iv) and $l = 2$ for the other parts. (The fact that this map factors through $\bar{R}_{l\delta}$ follows easily from the description of \mathcal{E}_d in §3.3.) In particular, it follows from the definitions that $\tau_r = \zeta_{r,2}(\bar{e}_\delta^{\otimes r-1} \otimes \tau_1 \otimes \bar{e}_\delta^{\otimes d-r-1})$.

Due to Lemma 6.10(ii), we have $e_{\delta d} \bar{R}_{d\delta} e_{\delta d} / \mathcal{J} = \bigoplus_{m \geq 0} T(F[z_1, \dots, z_d]_{\{2m\}})$. Define the linear map $\pi_m: e_{\delta d} \bar{R}_{d\delta} e_{\delta d} / \mathcal{J} \rightarrow e_{\delta d} \bar{R}_{d\delta} e_{\delta d} / \mathcal{J}$ as the projection onto the component $T(F[z_1, \dots, z_d]_{\{2m\}})$. We write $\tau = \tau_1$, $\epsilon = \epsilon_1$ and $\sigma = \sigma_1$.

Proof of Theorem 6.14(iv). We include only the proof of the first equality in part (iv), as the proof of the second one is similar. Due to Remark 6.15, we may (and do) assume that $d = 2$ and $r = 1$. First, we consider the case when $e > 2$. Due to Proposition 5.4 and Lemma 5.5(ii), it is enough to prove the desired equality for all $x \in \mathcal{B}_0 \sqcup \mathcal{B}_1$, where $\mathcal{B}_0, \mathcal{B}_1$ are as in Proposition 5.4. Thus, we may assume that $x = \psi_{w_{j,i}} e(\mathbf{i})$ for some $\mathbf{i}, \mathbf{j} \in I^{\otimes,1}$ with $j_e \in \{i_e, i_e + 1, i_e - 1\}$. Note that $x = \varphi_{w_{j,i}} e(\mathbf{i})$ because the entries of \mathbf{i} are pairwise distinct (cf. (6.1)). We view $w_{j,i}$ as an element of \mathfrak{S}_{2e} via the embedding $\mathfrak{S}_e \rightarrow \mathfrak{S}_{2e}$, $s_k \mapsto s_k$,

and we denote by $w'_{j,i}$ the image of $w_{j,i}$ under the embedding $\mathfrak{S}_e \rightarrow \mathfrak{S}_{2e}$, $s_k \mapsto s_{e+k}$ (for $1 \leq k < e$). By definitions of φ_{w_1} and η_t , we have $\eta_1(e(\mathbf{i}))\varphi_{w_1} = \varphi_{w_1}\eta_2(e(\mathbf{i}))$. Since $w_{j,i}w_1 = w_1w'_{j,i}$ and $w_1 \in {}^{(e,e)}\mathcal{D}_{2e}^{(e,e)}$, using (6.2), we obtain

$$\eta_1(x)\varphi_{w_1}e_{\delta^d} = \varphi_{w_{j,i}}\eta_1(e(\mathbf{i}))\varphi_{w_1} = \varphi_{w_1}\varphi_{w'_{j,i}}\eta_2(e(\mathbf{i})) = \varphi_{w_1}\eta_2(x)e_{\delta^d}.$$

Using (6.9), we deduce the following equalities in $\overline{R}_{2\delta}/\mathcal{J}$:

$$\begin{aligned} \eta_1(x) (\tau(z_1 - z_2)^e + \epsilon(z_1 - z_2)^{e-1}) &= \tau(z_1 - z_2)^e \eta_2(x) + \epsilon(z_1 - z_2)^{e-1} \eta_2(x) \\ (6.11) \qquad \qquad \qquad &= \tau \eta_2(x) (z_1 - z_2)^e + \epsilon \eta_2(x) (z_1 - z_2)^{e-1}, \end{aligned}$$

where the second equality follows from relations (3.8), (3.9), (3.10).

We claim that $\pi_e(\eta_1(x)\epsilon(z_1 - z_2)^{e-1}) = 0$. If $\deg(x) = 0$, then $\eta_1(x)\epsilon \in \iota(S_{\{2\}})$, and the claim follows. If $\deg(x) = 1$, then, using (6.10) and Corollary 6.6, we have

$$(6.12) \qquad \eta_1(x)\epsilon \in \eta_1((R_{\delta}^{\Lambda_0})_{\{2\}})\eta_2((R_{\delta}^{\Lambda_0})_{\{1\}}) \subset \mathcal{K},$$

where the second inclusion follows from the fact that $\eta((R_{\delta}^{\Lambda_0})_{\{2\}}) = F\{\psi_{w_{j,i}}(y_e - y_1)e(\mathbf{i}) \mid \mathbf{i}, \mathbf{j} \in I^{\emptyset,1}\}$. This concludes the proof of the claim. A similar argument shows that $\pi_e(\epsilon\eta_2(x)(z_1 - z_2)^{e-1}) = 0$. Hence, applying π_e to Equation (6.11) and using Lemma 6.10(ii), we obtain

$$\pi_0(\eta_1(x)\tau + \mathcal{J}) = \pi_0(\tau\eta_2(x) + \mathcal{J}).$$

By Corollary 6.11, we have $\eta_1(x)\tau, \tau\eta_2(x) \in T$. Thus, $\eta_1(x)\tau = \tau\eta_2(x)$ by Lemma 6.10(ii).

Now assume that $e = 2$. This case is treated by a direct calculation, as follows. First, note that $e_{\delta^2} = e(0101)$ and that

$$(6.13) \qquad \psi_1 e_{\delta^2} = 0 = \psi_3 e_{\delta^2}.$$

Indeed, $\psi_3 e(0101) = e(0110)\psi_3 = 0$ because $(0110) \notin \mathcal{E}_2$, and the other equality is proved similarly. We have

$$\begin{aligned} \varphi_{w_1} e_{\delta^2} &= \varphi_2 \varphi_1 \varphi_3 \varphi_2 e(0101) = \psi_2 \varphi_1 e(0011) \varphi_3 \varphi_2 = \psi_2 (\psi_1 (y_1 - y_2) + 1) e(0011) \varphi_3 \varphi_2 \\ (6.14) \qquad &= \psi_2 \psi_1 \varphi_3 e(0011) \varphi_2 (y_1 - y_3) + \psi_2 \varphi_3 e(0011) \varphi_2, \end{aligned}$$

where the last equality is due to Lemma 6.1(iii). Further,

$$\varphi_3 e(0011) \varphi_2 = (\psi_3 (y_3 - y_4) + 1) \varphi_2 e(0101) = \psi_3 \psi_2 e(0101) (y_2 - y_4) + \psi_2 e(0101).$$

Substituting this into (6.14), we obtain

$$\varphi_{w_1} e_{\delta^2} = (\psi_2 \psi_1 \psi_3 \psi_2 (y_1 - y_3) (y_2 - y_4) + \psi_2 \psi_1 \psi_2 (y_1 - y_3) + \psi_2 \psi_3 \psi_2 (y_2 - y_4) + \psi_2^2) e_{\delta^2}.$$

Since $\psi_2^2 e_{\delta^2} = -(y_2 - y_3)^2 e_{\delta^2}$ by (3.11), we deduce that

$$(6.15) \qquad \varphi_{w_1} e_{\delta^2} + \mathcal{J} = (\sigma - 1)(z_1 - z_2)^2 + (\psi_2 \psi_1 \psi_2 + \psi_2 \psi_3 \psi_2)(z_1 - z_2).$$

Now by the braid relations (3.12) and by (6.13), we have

$$\psi_2 \psi_1 \psi_2 e_{\delta^2} + \mathcal{J} = (\psi_1 \psi_2 \psi_1 + 2y_2 - y_1 - y_3) e_{\delta^2} + \mathcal{J} = z_1 - z_2.$$

Similarly, $\psi_2 \psi_3 \psi_2 e_{\delta^2} + \mathcal{J} = z_1 - z_2$. Substituting the last two identities into (6.15), we obtain $\varphi_{w_1} e_{\delta^2} + \mathcal{J} = (\sigma + 1)(z_1 - z_2)^2$, whence $\tau = (\sigma + 1)e_{\delta^2}$.

By Proposition 5.4, it is enough to show that $\eta_1(x)\tau = \tau\eta_2(x)$ for each $x \in \{e_{\delta^2}, y_2 e_{\delta^2}\}$. For $x = e_{\delta^2}$, this is clear, whereas for $x = y_2 e_{\delta^2}$, we have

$$\begin{aligned} \eta_1(x)\tau &= (y_2 - y_1)(\psi_2 \psi_1 \psi_3 \psi_2 + 1) e_{\delta^2} \\ &= \psi_2 \psi_1 y_3 \psi_3 e(0011) \psi_2 - \psi_2 y_1 \psi_1 e(0011) \psi_3 \psi_2 + (y_2 - y_1) e(0101) \\ &= (\psi_2 \psi_1 (\psi_3 y_4 - 1) \psi_2 - \psi_2 (\psi_1 y_2 - 1) \psi_3 \psi_2 + y_2 - y_1) e(0101) \\ &= (\psi_2 \psi_1 \psi_3 \psi_2 y_4 - \psi_2 \psi_1 \psi_2 - \psi_2 \psi_1 \psi_3 \psi_2 y_3 + \psi_2 \psi_3 \psi_2 + y_2 - y_1) e(0101) \end{aligned}$$

$$\begin{aligned}
&= (\sigma(y_4 - y_3) + y_2 - y_1 - (\psi_1\psi_2\psi_1 + 2y_2 - y_1 - y_3) \\
&\quad + (\psi_3\psi_2\psi_3 - 2y_3 + y_2 + y_4))e(0101) \\
&= (\sigma + 1)(y_4 - y_3)e(0101) = \tau\eta_2(x)
\end{aligned}$$

by (3.8), (3.9), (3.10), (3.12) and (6.13). \square

Lemma 6.16. *The set \mathcal{J} is a 2-sided ideal of $e_{\delta d}\overline{R}_{d\delta}e_{\delta d}$.*

Proof. Clearly, it suffices to show that $\mathcal{K}e_{\delta d}\overline{R}_{d\delta}e_{\delta d} \subset \mathcal{J}$. By Lemma 6.9(i), it is enough to prove that $\mathcal{K}\sigma_r e_{\delta d} \subset \mathcal{J}$ whenever $1 \leq r < d$ (as \mathcal{K} is a 2-sided ideal of $\overline{R}_{\delta d}$). It follows from Proposition 5.4 that \mathcal{K} is generated, as a left ideal of $\overline{R}_{\delta d}$, by the set $\bigcup_{t=1}^d \eta_t((R_{\delta}^{\Lambda_0})_{\{2\}})$. Also, $\sigma_r e_{\delta d} \in \tau_r + \overline{R}_{\delta d}$. By Theorem 6.14(iv), we have $\eta_t((R_{\delta}^{\Lambda_0})_{\{2\}})\tau_r = \tau_r \eta_{s_r(t)}((R_{\delta}^{\Lambda_0})_{\{2\}}) \subset \mathcal{J}$ for $1 \leq t \leq d$, and the result follows. \square

Set $H = \iota \circ (\eta^{\otimes d}): (R_{\delta}^{\Lambda_0})^{\otimes d} \rightarrow \overline{R}_{\delta d}$, so that $H(x_1 \otimes \cdots \otimes x_d) = \eta_1(x_1) \cdots \eta_d(x_d)$ for all $x_1, \dots, x_d \in R_{\delta}^{\Lambda_0}$.

Lemma 6.17. (i) *The unital subalgebra of $e_{\delta d}\overline{R}_{d\delta}e_{\delta d}$ generated by $\tau_1, \dots, \tau_{d-1}$ is contained in $T_{\{0\}}$.*

(ii) *The unital subalgebra of $e_{\delta d}\overline{R}_{d\delta}e_{\delta d}$ generated by $\{\tau_1, \dots, \tau_{d-1}\} \cup \iota(S)$ is contained in $T + \mathcal{J}$.*

Proof. By Corollary 6.11, we have $\tau_{r_1} \cdots \tau_{r_m} \in (e_{\delta d}\overline{R}_{d\delta}e_{\delta d})_{\{0\}} \subset T_{\{0\}}$ for any $r_1, \dots, r_m \in \{1, \dots, d-1\}$, which proves (i).

Let $\mathcal{S} = H((R_{\delta}^{\Lambda_0})^{\otimes d})$. It follows from the definition of η that $\iota(S) \subset \mathcal{S}$. By Theorem 6.14(iv), $\mathcal{S}\tau_r = \tau_r \mathcal{S}$ for $1 \leq r < d$. Hence, the subalgebra defined in (ii) is contained in the sum of the subspaces of the form $\tau_{r_1} \cdots \tau_{r_m} \mathcal{S}$ with $r_1, \dots, r_m \in \{1, \dots, d-1\}$. We have

$$(6.16) \quad T_{\{0\}}\iota(S) = \sum_{u \in \mathfrak{S}_d} \sigma_u \iota(S_{\{0\}})\iota(S) = \sum_{u \in \mathfrak{S}_d} \sigma_u \iota(S) = T,$$

where the first equality holds by Lemma 6.9(i). For $1 \leq r \leq d$, we have $\eta_r((R_{\delta}^{\Lambda_0})_{\{2\}}) \subset \mathcal{K}$ by Proposition 5.4. Hence, $\mathcal{S} \subset \iota(S) + \mathcal{K} \subset \iota(S) + \mathcal{J}$. Therefore,

$$\tau_{r_1} \cdots \tau_{r_m} \mathcal{S} \subset \tau_{r_1} \cdots \tau_{r_m} \iota(S) + \mathcal{J} \subset T_{\{0\}}\iota(S) + \mathcal{J} \subset T + \mathcal{J}. \quad \square$$

By Lemma 6.16, we have a natural algebra structure on $e_{\delta d}\overline{R}_{d\delta}e_{\delta d}/\mathcal{J}$.

Lemma 6.18. *For $1 \leq t \leq d$, the equality $z_t \varphi_{w_r} e_{\delta d} = \varphi_{w_r} z_{s_r(t)} e_{\delta d}$ holds in $e_{\delta d}\overline{R}_{d\delta}e_{\delta d}/\mathcal{J}$.*

Proof. We have

$$z_t \varphi_{w_r} e_{\delta d} = y_{(t-1)e+1} \varphi_{w_r} e_{\delta d} + \mathcal{J} = \varphi_{w_r} y_{(s_r(t)-1)e+1} e_{\delta d} + \mathcal{J} = \varphi_{w_r} z_{s_r(t)} e_{\delta d},$$

where the second equality is due to Lemma 6.1(iii). \square

Proof of Theorem 6.14 (i), (ii), (iii), (v). Part (iii) is clear from the definitions. For the remaining parts, we will repeatedly use Lemma 6.18 without explicit reference. By Remark 6.15, we may assume that $r = 1$. We consider part (i). By Lemma 6.1(v),

$$(6.17) \quad \varphi_{w_1}^2 e_{\delta d} + \mathcal{J} = (-1)^e (z_1 - z_2)^{2e}.$$

On the other hand, by (6.9),

$$\begin{aligned}
(6.18) \quad \varphi_{w_1}^2 e_{\delta d} + \mathcal{J} &= (\tau(z_1 - z_2)^e + \epsilon(z_1 - z_2)^{e-1})\varphi_{w_1} \\
&= (-1)^e \tau \varphi_{w_1} (z_1 - z_2)^e + (-1)^{e-1} \epsilon \varphi_{w_1} (z_1 - z_2)^{e-1} \\
&= (-1)^e \tau^2 (z_1 - z_2)^{2e} + (-1)^e \tau \epsilon (z_1 - z_2)^{2e-1} \\
&\quad + (-1)^{e-1} \epsilon \tau (z_1 - z_2)^{2e-1} + (-1)^{e-1} \epsilon^2 (z_1 - z_2)^{2e-2}.
\end{aligned}$$

By Lemma 6.17(ii), we have $\epsilon\tau, \tau\epsilon, \epsilon^2 \in T + \mathcal{J}$. Therefore, applying π_{2e} to the right-hand sides of (6.17) and (6.18), we obtain $(z_1 - z_2)^{2e} = \tau^2(z_1 - z_2)^{2e}$. By Lemma 6.17(i), we have $\tau^2 \in T$, whence $\tau^2 = e_{\delta d}$ by Lemma 6.10(ii).

For (ii), writing $\nu_0 = \tau_1$, $\nu_1 = \epsilon_1$, $\nu'_0 = \tau_2$, $\nu'_1 = \epsilon_2$, we have

$$\begin{aligned} \varphi_{w_1}\varphi_{w_2}\varphi_{w_1}e_{\delta d} + \mathcal{J} &= \sum_{a \in \{0,1\}} \nu_a(z_1 - z_2)^{e-a} \varphi_{w_2}\varphi_{w_1} \\ &= \sum_{a \in \{0,1\}} \nu_a \varphi_{w_2}\varphi_{w_1}(z_2 - z_3)^{e-a} \\ &= \sum_{a,b \in \{0,1\}} \nu_a \nu'_b (z_2 - z_3)^{e-b} \varphi_{w_1}(z_2 - z_3)^{e-a} \\ &= \sum_{a,b \in \{0,1\}} \nu_a \nu'_b \varphi_{w_1}(z_1 - z_3)^{e-b} (z_2 - z_3)^{e-a} \\ &= \sum_{a,b,c \in \{0,1\}} \nu_a \nu'_b \nu'_c (z_1 - z_2)^{e-c} (z_1 - z_3)^{e-b} (z_2 - z_3)^{e-a}. \end{aligned}$$

By Lemma 6.17(ii), we have $\nu_a \nu'_b \nu'_c \in T + \mathcal{J}$ in all cases. Hence, π_{3e} fixes the summand corresponding to $a = b = c = 0$ and sends the other 7 summands to 0, so

$$\pi_{3e}(\varphi_{w_1}\varphi_{w_2}\varphi_{w_1}e_{\delta d} + \mathcal{J}) = \tau_1\tau_2\tau_1(z_1 - z_2)^e(z_1 - z_3)^e(z_2 - z_3)^e.$$

A similar computation yields

$$\pi_{3e}(\varphi_{w_2}\varphi_{w_1}\varphi_{w_2}e_{\delta d} + \mathcal{J}) = \tau_2\tau_1\tau_2(z_1 - z_2)^e(z_1 - z_3)^e(z_2 - z_3)^e.$$

By Lemma 6.1(ii), we have $\varphi_{w_1}\varphi_{w_2}\varphi_{w_1} = \varphi_{w_2}\varphi_{w_1}\varphi_{w_2}$. By Lemmas 6.17(i) and 6.10(ii), the equality $\tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2$ follows.

For (v), we compute

$$\varphi_{w_1}z_1\varphi_{w_1} = \varphi_{w_1}^2 z_2 = (-1)^e(z_1 - z_2)^{2e} z_2$$

using (6.17) and, on the other hand,

$$\begin{aligned} \varphi_{w_1}z_1\varphi_{w_1} &= (\tau(z_1 - z_2)^e + \epsilon(z_1 - z_2)^{e-1})z_1\varphi_{w_1} \\ &= \tau z_1\varphi_{w_1}(z_2 - z_1)^e + \epsilon\varphi_{w_1}z_2(z_2 - z_1)^{e-1} \\ &= (-1)^e\tau y_1\tau(z_1 - z_2)^{2e} + (-1)^e\tau z_1\epsilon(z_1 - z_2)^{2e-1} + \epsilon\varphi_{w_1}z_2(z_2 - z_1)^{e-1} \\ &= (-1)^e\tau y_1\tau(z_1 - z_2)^{2e} + (-1)^e\tau z_1\epsilon(z_1 - z_2)^{2e-1} \\ &\quad + (-1)^{e-1}\epsilon\tau z_2(z_1 - z_2)^{2e-1} + (-1)^{e-1}\epsilon^2 z_2(z_1 - z_2)^{2e-2}. \end{aligned}$$

Hence, using Lemma 6.17(ii) (and the equality $z_1\epsilon = \epsilon z_1$), we obtain

$$(-1)^e(z_1 - z_2)^{2e} z_2 = \pi_{2e+1}(\varphi_{w_1}z_1\varphi_{w_1}) = \pi_{2e+1}((-1)^e\tau y_1\tau(z_1 - z_2)^{2e}),$$

whence, by Lemma 6.10(ii), we have $z_2 = \pi_1(\tau y_1\tau + \mathcal{J})$. Since $\deg(\tau y_1\tau) = 2$, it follows that $\tau y_1\tau + \mathcal{J} = z_2 + (x + \mathcal{J})$ for some $x \in T_{\{2\}}$. Hence, $y_{e+1} - \tau y_1\tau \in T + \mathcal{J}$. By Lemma 6.9(i), $\mathcal{J} = \sum_{u \in \mathfrak{S}_d} \sigma_u \mathcal{K} \subset T\mathcal{K}$, and (v) follows. \square

Write $I^{(\emptyset,1)^d} = \{(\mathbf{i}^{(1)} \dots \mathbf{i}^{(d)}) \mid \mathbf{i}^{(1)}, \dots, \mathbf{i}^{(d)} \in I^{\emptyset,1}\}$. Note that $e_{\delta d} = \sum_{\mathbf{i} \in I^{(\emptyset,1)^d}} e(\mathbf{i})$ by Lemma 4.5. Recall that $(R_{\delta}^{\Lambda_0})^{\otimes d}$ and $F\mathfrak{S}_d$ are identified with subalgebras of $R^{\Lambda_0} \wr \mathfrak{S}_d$, and recall the homomorphism $H: (R_{\delta}^{\Lambda_0})^{\otimes d} \rightarrow \overline{R}_{\delta d}$ defined before Lemma 6.17.

Corollary 6.19. (i) *There is a graded unital algebra homomorphism*

$$\Theta: R_{\delta}^{\Lambda_0} \wr \mathfrak{S}_d \rightarrow e_{\delta d} \overline{R}_{\delta d} e_{\delta d}$$

given by $\Theta(a) = H(a)$ and $\Theta(s_r) = \tau_r$ for all $a \in (R_{\delta}^{\Lambda_0})^{\otimes d}$ and $1 \leq r < d$.

(ii) Suppose that $\omega: e_{\delta d} \bar{R}_{d\delta} e_{\delta d} \rightarrow A$ is a unital algebra homomorphism onto an algebra A such that for all $\mathbf{i} \in I^{(\emptyset, 1)^d}$ we have $\omega(y_1 e(\mathbf{i})) \in F\{\omega((y_e - y_1)e(\mathbf{i}))\}$. Then the composition $\omega \circ \Theta: R_{\delta}^{\Lambda_0} \wr \mathfrak{S}_d \rightarrow A$ is surjective.

Proof. The fact that Θ is a homomorphism follows directly from Theorem 6.14(i)–(iv). Since H is graded and $\deg(s_r) = 0 = \deg(\tau_r)$ for $1 \leq r < d$, we see that Θ is graded as well.

For (ii), observe that $\iota(S) \subset \text{im } H \subset \text{im } \Theta$, whence $\sigma_r e_{\delta d} \in \tau_r + \iota(S) \subset \text{im } \Theta$ for $1 \leq r < d$. By Lemma 6.9(i) and Corollary 6.6, the algebra $e_{\delta d} \bar{R}_{d\delta} e_{\delta d}$ is generated by the subset $T \cup F[y_1, \dots, y_{de}]e_{\delta d}$. Further, $T = \sum_{u \in \mathfrak{S}_d} \sigma_u \iota(S) \subset \text{im } \Theta$ (cf. (6.16)), so it is enough to show that $\omega(y_l e_{\delta d}) \in \text{im}(\omega \circ \Theta)$ whenever $1 \leq l \leq de$. Note also that $\mathcal{K} = \mathcal{K}_{\geq 2}$ and $\mathcal{K}_{\{2\}} \subset \text{im } H$ because $\mathcal{K}_{\{2\}}$ is generated by $\iota(S)_{\{0\}}$ and the elements of the form $(y_t - y_{t'})e(\mathbf{i})$ with $t, t' \in \{(k-1)e+1, \dots, ke\}$ for some $k \in \{1, \dots, d\}$ and $\mathbf{i} \in I^{(\emptyset, 1)^d}$, and all such elements $(y_t - y_{t'})e(\mathbf{i})$ belong to $\text{im } H$ by definition of η .

Let \mathcal{T} be defined as in Theorem 6.14(v); then $\mathcal{T}_{\{2\}} \subset \text{im } \Theta$. Now we prove by induction on r that $\omega(y_{re+1} e_{\delta d}) \in \text{im}(\omega \circ \Theta)$ for all $r = 1, \dots, d$. The case $r = 1$ holds by the hypothesis, and the inductive step follows from the fact that $y_{re+1} e_{\delta d} - \tau_r y_{(r-1)e+1} \tau_r \in \mathcal{T}_{\{2\}}$ for $1 \leq r < d$, which is derived from Theorem 6.14(v) by considering degrees. Since $\mathcal{K}_{\{2\}} \subset \text{im } H$, it follows that $\omega(y_l e_{\delta d}) \in \text{im}(\omega \circ \Theta)$ for all l , as claimed. \square

7. SURJECTIVITY OF THE HOMOMORPHISM

Let $\mathcal{H}_{\rho, d}$ be a RoCK block of residue κ . We have constructed all the maps in the diagram (3.26): the homomorphism Ω is defined after Proposition 4.11, and Θ is defined by Corollary 6.19(i). Thus, we have a graded unital algebra homomorphism $\Xi = \Xi^{(d)} := \Omega \circ \text{rot}_{\kappa} \circ \Theta: R_{\delta}^{\Lambda_0} \wr \mathfrak{S}_d \rightarrow C_{\rho, d}$. Due to Proposition 4.12, in order to complete the proof of Theorem 3.4, we only need to show that Ξ is surjective. First, we state and prove Proposition 7.2, which applies to the case when $d = 1$. Using Corollary 6.19(ii), we will then deduce surjectivity of Ξ in the general case in §7.3.

7.1. The case $d = 1$. In this subsection, we assume that ρ is a Rouquier e -core of residue κ for the integer 1, write $C = C_{\rho, 1}$ and consider the homomorphism $\Xi := \Xi^{(1)}: R_{\delta}^{\Lambda_0} \rightarrow C$. By Proposition 4.12, we have $\text{qdim } C = \text{qdim}(R_{\delta}^{\Lambda_0})$, and hence $C = C_{\{0, 1, 2\}}$ by Proposition 5.4.

For $\mathbf{i} \in I^e$, define $e'(\mathbf{i}) = \sum_{j \in I^{\rho, 0}} e(j\mathbf{i}) \in R_{|\rho|+e}^{\Lambda_0}$. Set

$$f = \text{BK}_{|\rho|+e}(f_{\rho, 1}) = \sum_{j \in I^{\rho, 0}, \mathbf{i} \in I_{+\kappa}^{\emptyset, 1}} e(j\mathbf{i}) \in R_{|\rho|+e}^{\Lambda_0};$$

the second equality follows from (3.21) and Proposition 4.11(ii). By the definitions, the map $\Xi: R_{\delta}^{\Lambda_0} \rightarrow C_{\rho, 1}$ is given by

$$(7.1) \quad e(\mathbf{i}) \mapsto e'(\mathbf{i}^{+\kappa}), \quad \psi_r \mapsto \psi_{|\rho|+r} f, \quad y_t \mapsto (y_{|\rho|+t} - y_{|\rho|+1}) f$$

for $\mathbf{i} \in I^{\delta}$, $1 \leq r < e$ and $1 \leq t \leq e$. By Proposition 4.11, the algebra C is generated by the set

$$\{e'(\mathbf{i}^{+\kappa}) \mid \mathbf{i} \in I^{\emptyset, 1}\} \cup \{\psi_{|\rho|+r} f \mid 1 \leq r < e\} \cup \{y_{|\rho|+r} f \mid 1 \leq r \leq e\}.$$

It follows that Ξ restricts to a surjective vector space homomorphism from $(R_{\delta}^{\Lambda_0})_{<2}$ onto $C_{<2}$, which is seen to be an isomorphism by comparing dimensions. In particular, $R_{\delta}^{\Lambda_0} / (R_{\delta}^{\Lambda_0})_{\{2\}} \cong C / C_{\{2\}}$ as graded algebras.

Lemma 7.1. *For all $\mathbf{i} \in I^{\emptyset, 1}$, we have $\dim(e'(\mathbf{i}^{+\kappa}) C_{\{2\}} e'(\mathbf{i}^{+\kappa})) = 1$.*

Proof. Recall the notation introduced before Lemma 4.13. By Theorem 4.1 and Lemma 4.3, $\text{qdim}(e'(\mathbf{i}^{+\kappa}) R_{\text{cont}(\rho)+\delta}^{\Lambda_0} e'(\mathbf{i}^{+\kappa})) = \sum_{\mathbf{t}, \mathbf{u}} q^{\deg(\mathbf{t}) + \deg(\mathbf{u})}$ where the sum is over all pairs $(\mathbf{t}, \mathbf{u}) \in$

$\text{Std}'_e(\rho, d)^{\times 2}$ such that $\text{Shape}(\mathbf{t}) = \text{Shape}(\mathbf{u})$ and, if $\beta(\mathbf{t}) = (\mathbf{t}_{\leq |\rho|}, \mathbf{s})$ and $\beta(\mathbf{u}) = (\mathbf{u}_{\leq |\rho|}, \mathbf{r})$, then $\mathbf{i}^{\mathbf{s}} = \mathbf{i}^{\mathbf{r}} = \mathbf{i}$. By Lemmas 4.13 and 4.14 (together with Theorem 4.1), this sum is equal to $\text{qdim}(R_{\text{cont}(\rho)}^{\Lambda_0}) \sum_{\mathbf{s}, \mathbf{r}} q^{\deg(\mathbf{s}) + \deg(\mathbf{r})}$, where the sum is over all pairs (\mathbf{s}, \mathbf{r}) of standard tableaux such that $\mathbf{i}^{\mathbf{s}} = \mathbf{i}^{\mathbf{r}} = \mathbf{i}$ and $\text{Shape}(\mathbf{s}) = \text{Shape}(\mathbf{r})$. Dividing by $\text{qdim}(R_{\text{cont}(\rho)}^{\Lambda_0})$ and using the isomorphism (4.8), we deduce that $\text{qdim}(e'(\mathbf{i}^{+\kappa})Ce'(\mathbf{i}^{+\kappa})) = \sum_{\mathbf{s}, \mathbf{r}} q^{\deg(\mathbf{s}) + \deg(\mathbf{r})}$, and the result follows by Lemma 5.2. \square

Proposition 7.2. *For every $\mathbf{i} \in I^{\varnothing, 1}$, we have $(y_{|\rho|+e} - y_{|\rho|+1})e'(\mathbf{i}^{+\kappa}) \neq 0$ in $R_{\text{cont}(\rho)+\delta}^{\Lambda_0}$, and hence $y_{|\rho|+1}e'(\mathbf{i}^{+\kappa}) \in F(y_{|\rho|+e} - y_{|\rho|+1})e'(\mathbf{i}^{+\kappa})$.*

If the first statement of the proposition holds, then, by Lemma 7.1, $e'(\mathbf{i}^{+\kappa})C_{\{2\}}e'(\mathbf{i}^{+\kappa}) = F(y_{|\rho|+e} - y_{|\rho|+1})e'(\mathbf{i}^{+\kappa})$ for any $\mathbf{i} \in I^{\varnothing, 1}$, and the second statement of the proposition follows. So we only need to prove the first statement.

Proof of Proposition 7.2 for $e \geq 3$. It is well known that the algebra $R_{d+e}^{\Lambda_0} \cong \mathcal{H}_{|\rho|+e}$ is symmetric (see e.g. [31, Corollary V.5.4]). Hence, writing $f = \text{BK}_{|\rho|+e}(f_{\rho, 1})$, we see that $fR_{\text{cont}(\rho)+\delta}^{\Lambda_0}f = fR_{|\rho|+e}^{\Lambda_0}f$ is symmetric as well by [31, Theorem IV.4.1]. Further, by Proposition 4.9 and the isomorphism (4.8), the algebra $C := C_{\rho, 1}$ is Morita equivalent to $fR_{\text{cont}(\rho)+\delta}^{\Lambda_0}f$, whence C is also symmetric by [31, Corollary IV.4.3]. Thus, C is isomorphic to $C^* := \text{Hom}_F(C, F)$ as a (C, C) -bimodule. Note that we have a grading on C^* defined in the usual way: for $n \in \mathbb{Z}$ and $\xi \in C^*$, we have $\xi \in C_{\{n\}}^*$ if and only if $\xi|_{C_{\mathbb{Z} \setminus \{-n\}}} = 0$.

Note that C has only one block, i.e. is indecomposable as a (C, C) -bimodule. This follows, for example, from the fact that the indecomposable algebra $\mathcal{H}_{\rho, 1}$ is Morita equivalent to $f_{\rho, 1}\mathcal{H}_{\rho, 1}f_{\rho, 1}$ by Proposition 8.2((ii) \Leftrightarrow (iii)) proved below and hence is Morita equivalent to C . (Alternatively, using Proposition 5.4, one can show without difficulty that $Z(R_{\delta}^{\Lambda_0}/(R_{\delta}^{\Lambda_0})_{\{2\}})_{\{0\}}$ is 1-dimensional, so $Z(C)_{\{0\}}$ must also be 1-dimensional because $C/C_{\{2\}} \cong R_{\delta}^{\Lambda_0}/(R_{\delta}^{\Lambda_0})_{\{2\}}$.) By [1, Lemma 2.5.3], it follows that $C^* \cong C\langle n \rangle$ as graded (C, C) -bimodules for some $n \in \mathbb{Z}$. Since $C = C_{\{0, 1, 2\}}$ with $C_{\{2\}} \neq 0$ and $C^* = C_{\{0, -1, -2\}}^*$, we have $n = -2$. The graded isomorphism $C \xrightarrow{\sim} C^*\langle 2 \rangle$ of (C, C) -bimodules sends 1 to some element $\xi \in (C^*)_{\{-2\}}$ such that the bilinear form $C \times C \rightarrow F$ given by $(a, b) \mapsto \xi(ab)$ is symmetric and non-degenerate.

Let $\mathbf{i} \in I^{\varnothing, 1}$. Since $e \geq 3$, we can find $\mathbf{j} \in I^{\varnothing, 1}$ such that $j_e \in \{i_e + 1, i_e - 1\}$. Then $\psi_{w_{i, j}}e(\mathbf{j})$ is a non-zero element of $(R_{\delta}^{\Lambda_0})_{\{1\}}$ by Proposition 5.4. Hence, $\Xi(\psi_{w_{i, j}}e(\mathbf{j})) \neq 0$, so there exists $a \in C_{\{1\}}$ such that $\xi(\Xi(\psi_{w_{i, j}}e(\mathbf{j}))a) \neq 0$. Since the bilinear form in question is symmetric and $\Xi(\psi_{w_{i, j}}e(\mathbf{j})) = e'(\mathbf{i}^{+\kappa})\Xi(\psi_{w_{i, j}}e(\mathbf{j}))$, we have $\xi(\Xi(\psi_{w_{i, j}}e(\mathbf{j}))ae'(\mathbf{i}^{+\kappa})) \neq 0$, so we may assume that $a = e'(\mathbf{j}^{+\kappa})ae'(\mathbf{i}^{+\kappa})$. Now the subspace $e'(\mathbf{j}^{+\kappa})C_{\{1\}}e'(\mathbf{i}^{+\kappa}) = \Xi(e(\mathbf{j})(R_{\delta}^{\Lambda_0})_{\{1\}}e(\mathbf{i}))$ is 1-dimensional and is spanned by $\Xi(\psi_{w_{j, i}}e(\mathbf{i}))$ by Proposition 5.4. This implies that

$$0 \neq \Xi(\psi_{w_{i, j}}e(\mathbf{j}))\Xi(\psi_{w_{j, i}}e(\mathbf{i})) = \Xi(\psi_{w_{i, j}}\psi_{w_{j, i}}e(\mathbf{i})) = \pm \Xi(y_e e(\mathbf{i})) = \pm (y_{|\rho|+e} - y_{|\rho|+1})e'(\mathbf{i}^{+\kappa}),$$

where the penultimate equality holds by Lemma 5.5(i). \square

7.2. A calculation involving the Brundan–Kleshchev isomorphism. In order to prove Proposition 7.2 for $e = 2$, we will compute $\text{BK}_{|\rho|+e}(T_{|\rho|+1}f_{\rho, 1})$. For later use in Section 9, we begin with a more general set-up and for now let the integers $e \geq 2$ and $d \geq 1$ be arbitrary.

Choose and fix an integer N large enough so that for every $w \in \mathfrak{S}_{de}$ and $\mathbf{i} \in I^{d\delta}$ we have $\deg(\psi_w e(\mathbf{i})) + N > 2$. Let $\Lambda = N(\Lambda_0 + \cdots + \Lambda_{e-1}) \in P_+$. It follows from Theorem 3.5 that the 2-sided ideal of $R_{d\delta}$ generated by $\{y_t^N e_{d\delta} \mid 1 \leq t \leq de\}$ has a zero component in degree

m for all integers $m \leq 2$. This ideal is the kernel of the canonical projection $R_{d\delta} \rightarrow R_{d\delta}^\Lambda$, which therefore restricts to a vector space isomorphism on each m -component for $m \leq 2$. Let $\overline{R}_{d\delta}^\Lambda$ be the quotient of $R_{d\delta}^\Lambda$ by the two-sided ideal generated by $\{e(\mathbf{i}) \mid \mathbf{i} \in I^{d\delta} \setminus \mathcal{E}_d\}$ (cf. the definition of $\overline{R}_{d\delta}$ in §3.3). As usual, $\overline{}$ denotes the natural projections $R_{de} \rightarrow \overline{R}_{d\delta}$ and $R_{de}^\Lambda \rightarrow \overline{R}_{d\delta}^\Lambda$. As in §6.3, symbols that would ordinarily represent elements of R_{de} , will denote instead their images in $\overline{R}_{d\delta}$. In the rest of this subsection and in Section 9, we abuse notation by identifying $(R_{d\delta})_{\leq 2}$ with $(R_{d\delta}^\Lambda)_{\leq 2}$ and $(\overline{R}_{d\delta})_{\leq 2}$ with $(\overline{R}_{d\delta}^\Lambda)_{\leq 2}$. In particular, $\overline{\text{BK}_{de}^\Lambda(T_r)}_{\{0,1,2\}} e_{\delta d}$ is viewed as an element of $\overline{R}_{d\delta}$ for $1 \leq r < de$.

Lemma 7.3. *Assume that $e = 2$. If $1 \leq r \leq d$, we have*

$$\overline{\text{BK}_{2d}^\Lambda(T_{2r-1})}_{\{0,1,2\}} e_{\delta d} = \begin{cases} (1 + y_{2r-1} - y_{2r}) e_{\delta d} & \text{if } \text{char } F = 2, \\ \left(-1 + \frac{y_{2r} - y_{2r-1}}{2}\right) e_{\delta d} & \text{if } \text{char } F \neq 2. \end{cases}$$

Proof. By Lemma 6.3 and the statements after Lemma 6.8, $\psi_{2r-1} e_{\delta d} = 0$. Note that $e_{\delta d} = e(0101 \dots 01)$. Hence, by (3.14), we have $\overline{\text{BK}_{2d}^\Lambda(T_{2r-1})}_{\{0,1,2\}} e_{\delta d} = -P_1(y_{2r-1}, y_{2r}) e_{\delta d}$. Using the formulas (3.16) and (3.18) for P_1 , one concludes the proof by an easy calculation (note that $\xi = -1$ if $\text{char } F \neq 2$). \square

Proof of Proposition 7.2 for $e = 2$. As in the statement of the proposition, we consider the case when $d = 1$ and assume all the notation of §7.1. Note that $\mathbf{i} = (01)$, so $\mathbf{i}^{+\kappa}$ is either (01) or (10) , and $f := \text{BK}_{|\rho|+e}(f_{\rho,1}) = e'(\mathbf{i}^{+\kappa})$. It follows from (3.14) and the definitions of Ω and rot_κ that $\text{BK}_{|\rho|+2}(T_{|\rho|+1})_{\{0,1,2\}} f = \Omega(\text{rot}_\kappa(\overline{\text{BK}_2^\Lambda(T_1)}_{\{0,1,2\}} e_\delta))$ and that $\text{BK}_{|\rho|+2}(T_{|\rho|+1}) f$ belongs to the unital subalgebra of $fR_{|\rho|+2}^{\Lambda_0} f$ generated by the set $\{y_{|\rho|+1} f, y_{|\rho|+2} f\}$ and hence to C (note that we have $\psi_{|\rho|+1} f = \text{rot}_\kappa(\Omega(\psi_1 e_\delta)) = 0$). Since $C = C_{\{0,1,2\}}$, we see that $\text{BK}_{|\rho|+2}(T_{|\rho|+1}) f = \Omega(\text{rot}_\kappa(\overline{\text{BK}_2^\Lambda(T_1)}_{\{0,1,2\}} e_\delta))$. Assume for contradiction that $(y_{|\rho|+2} - y_{|\rho|+1}) f = 0$. Using Lemma 7.3, we deduce that $\text{BK}_{|\rho|+2}(T_{|\rho|+1}) f = \Omega(\text{rot}_\kappa(e_\delta)) = -f$. Hence, $T_{|\rho|+1} f_{\rho,1} = -f_{\rho,1}$.

Note that $T_{|\rho|+1}$ commutes with $f_{\rho,1}$. Let us view $\mathcal{H}_{|\rho|+2}$ and $M := f_{\rho,1} \mathcal{H}_{|\rho|+2} f_{\rho,1}$ as \mathcal{H}_2 -modules with T_1 acting via left multiplication by $T_{|\rho|+1}$. It follows from the identity just proved that M is a direct sum of 1-dimensional simple \mathcal{H}_2 -modules. On the other hand, M must be projective because it is a direct summand of $\mathcal{H}_{|\rho|+2}$, which is a free \mathcal{H}_2 -module with basis $\{T_w \mid w \in {}^{(1|\rho|,2)}\mathcal{D}_{|\rho|+2}\}$. This is a contradiction because the only indecomposable projective \mathcal{H}_2 -module is \mathcal{H}_2 itself (note that \mathcal{H}_2 is isomorphic to the truncated polynomial algebra $F[x]/(x^2)$). \square

7.3. Conclusion of the proof of Theorem 3.4. We return to the case when $d \geq 1$ is arbitrary and $\mathcal{H}_{\rho,d}$ is a RoCK block of residue κ . As usual, write $f = \text{BK}_{|\rho|+de}(f_{\rho,d})$. Let $\mathbf{i} \in I^{(\emptyset,1)^d}$ (see the definition before Corollary 6.19), and write $e'(\mathbf{i}) = \sum_{\mathbf{j} \in I^{\rho,0}} e(\mathbf{j}\mathbf{i}) \in fR_{|\rho|+de}^{\Lambda_0} f$. For $1 \leq r \leq e$, we have $(\Omega \circ \text{rot}_\kappa)(y_r e(\mathbf{i})) = y_{|\rho|+r} e'(\mathbf{i}^{+\kappa})$. Applying the map $x \mapsto \iota_{|\rho|+e}^{|\rho|+de}(x) e'(\mathbf{i})$ to the second statement of Proposition 7.2, we see that $y_{|\rho|+1} e'(\mathbf{i}^{+\kappa}) \in F(y_{|\rho|+e} - y_{|\rho|+1}) e'(\mathbf{i}^{+\kappa})$. Hence, by Corollary 6.19(ii), the graded algebra homomorphism $\Xi: R_\delta^{\Lambda_0} \wr \mathfrak{S}_d \rightarrow C_{\rho,d}$ is surjective, whence it is an isomorphism by Proposition 4.12. The proof of Theorem 3.4 is complete.

8. TWO OBSERVATIONS

8.1. Another formula for the idempotent $f_{\rho,d}$. Let ρ be a Rouquier core for an integer $d \geq 0$. If \mathcal{O} is an integral domain, $t \in \mathcal{O}^\times$ and $1 \leq r \leq d$, let $\beta_r: \mathcal{H}_e(\mathcal{O}, t) \rightarrow \mathcal{H}_{|\rho|+ed}(\mathcal{O}, t)$ be the unital algebra homomorphism defined by $T_j \mapsto T_{|\rho|+(r-1)e+j}$, $1 \leq j < e$. As before,

for $n \geq m \geq 0$, we view $\mathcal{H}_m(\mathcal{O}, t)$ as a subalgebra of $\mathcal{H}_n(\mathcal{O}, t)$ via the embedding $T_j \mapsto T_j$, $1 \leq j < m$.

If $1 \leq r \leq d$, let $b_{\emptyset,1}^{(r)} = \beta_r(b_{\emptyset,1})$. Note that the idempotents $b_{\emptyset,1}^{(1)}, \dots, b_{\emptyset,1}^{(d)}$ commute with $f_{\rho,d}$ and with each other. Turner [33, Chapter IV] considers the idempotent $f_{\rho,d} b_{\emptyset,1}^{(1)} \cdots b_{\emptyset,1}^{(d)}$ rather than $f_{\rho,d}$ (cf. also [6, Section 4]), but the following proposition shows that this makes no difference.

Proposition 8.1. *In any RoCK block $\mathcal{H}_{\rho,d}$, we have $f_{\rho,d} b_{\emptyset,1}^{(1)} \cdots b_{\emptyset,1}^{(d)} = f_{\rho,d}$.*

Proof. As in [11, Section 5], let $\mathcal{O} = F[t]_{(t-\xi)}$, the localisation of the polynomial ring $F[t]$ at the ideal $(t - \xi)F[t]$, and consider the field of fractions $K = F(t)$ of \mathcal{O} . For any $n \geq 0$, we have $\mathcal{H}_n \cong \mathcal{H}_n(\mathcal{O}, t)/(t - \xi)F[t]\mathcal{H}_n(\mathcal{O}, t)$. Since t is not a root of unity in K , the algebra $\mathcal{H}_n(K, t)$ is semisimple, with $\{S^{\lambda, K, t} \mid \lambda \in \text{Par}(n)\}$ being a complete set of non-isomorphic simple modules (see [11, Theorem 4.3]). For every partition λ of n , let b_λ be the primitive central idempotent of $\mathcal{H}_n(K, t)$ such that $b_\lambda S^{\lambda, K, t} = S^{\lambda, K, t}$. For any $(\pi, l) \in \text{Bl}_e(n)$, let $\tilde{b}_{\pi, l} = \sum_{\lambda \in \text{Par}_e(\pi, l)} b_\lambda$; further, let $\tilde{b}_{\emptyset,1}^{(r)} = \beta_r(\tilde{b}_{\emptyset,1})$. It follows from the results of [11, Section 5] together with [11, Lemma 4.6 and Theorem 4.7] that $\tilde{b}_{\pi, l} \in \mathcal{H}_n(\mathcal{O}, t)$ and that $b_{\pi, l}$ is the image of $\tilde{b}_{\pi, l}$ under the canonical projection $\mathcal{H}_n(\mathcal{O}, t) \rightarrow \mathcal{H}_n$; further, $b_{\emptyset,1}^{(r)}$ is the image of $\tilde{b}_{\emptyset,1}^{(r)}$ under the canonical map $\mathcal{H}_{|\rho|+de}(\mathcal{O}, t) \rightarrow \mathcal{H}_{|\rho|+de}$.

If λ, μ and ν are partitions with $|\lambda| = |\mu| + |\nu|$, let $c_{\mu\nu}^\lambda$ be the corresponding Littlewood–Richardson coefficient (see e.g. [14, Section 2.8]). It follows from [12, Proposition 13.7(iii)] that the Littlewood–Richardson rule holds for the algebras $\mathcal{H}_n(K, t)$: if λ, μ and ν are partitions such that $|\lambda| = |\mu| + |\nu|$ and if we identify $\mathcal{H}_{|\mu|}(K, t) \otimes_K \mathcal{H}_{|\nu|}(K, t)$ with the parabolic subalgebra generated by $\{T_j \mid 1 \leq j < |\mu| \text{ or } |\mu| < j < |\lambda|\}$ of $\mathcal{H}_{|\lambda|}(K, t)$ in the obvious way, then the $\mathcal{H}_{|\mu|}(K, t) \otimes_K \mathcal{H}_{|\nu|}(K, t)$ -module $S^{\mu, K, t} \otimes_K S^{\nu, K, t}$ appears in the restriction of $S^{\lambda, K, t}$ to $\mathcal{H}_{|\mu|}(K, t) \otimes_K \mathcal{H}_{|\nu|}(K, t)$ with multiplicity $c_{\mu\nu}^\lambda$. Let $1 \leq r \leq d$, $\mu \in \text{Par}_e(\rho, r-1)$, $\lambda \in \text{Par}_e(\rho, r)$ and $\nu \in \text{Par}(e) \setminus \text{Par}_e(\emptyset, 1)$. By Lemma 4.3 and the standard combinatorial description of the Littlewood–Richardson coefficients, we have $c_{\mu\nu}^\lambda = 0$, so $S^{\mu, K, t} \otimes_K S^{\nu, K, t}$ is not a summand of the restriction of $S^{\lambda, K, t}$ to $\mathcal{H}_{|\rho|+(r-1)e}(K, t) \otimes_K \mathcal{H}_e(K, t)$. Therefore, $b_\lambda b_\mu \beta_r(b_\nu) = 0$. Summing over all such λ, μ, ν , we deduce that $\tilde{b}_{\rho, r-1} \tilde{b}_{\rho, r} (1 - \tilde{b}_{\emptyset,1}^{(r)}) = 0$, whence $\tilde{b}_{\rho, r-1} \tilde{b}_{\rho, r} \tilde{b}_{\emptyset,1}^{(r)} = \tilde{b}_{\rho, r-1} \tilde{b}_{\rho, r}$. Applying the projection onto $\mathcal{H}_{|\rho|+de}$ and using the statements at the end of the previous paragraph, we see that $b_{\rho, r-1} b_{\rho, r} b_{\emptyset,1}^{(r)} = b_{\rho, r-1} b_{\rho, r}$. The result follows (cf. (1.5)). \square

8.2. A condition for Morita equivalence. Let $\mathcal{H}_{\rho,d}$ be a RoCK block.

Proposition 8.2. *The following are equivalent:*

- (i) $f_{\rho,d} D \neq 0$ for all simple $\mathcal{H}_{\rho,d}$ -modules D .
- (ii) $f_{\rho,d} \mathcal{H}_{\rho,d} f_{\rho,d}$ is Morita equivalent to $\mathcal{H}_{\rho,d}$.
- (iii) $d < \text{char } F$ or $\text{char } F = 0$.

If these statements hold, then $\mathcal{H}_{\rho,d}$ is Morita equivalent to $\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d$.

Proof. For a finite-dimensional algebra A , denote by $\ell(A)$ the number of isomorphism classes of simple A -modules. The equivalence between (i) and (ii) follows from the general properties of idempotent truncations recalled in §1.1. In view of those properties and Theorem 1.1, statement (ii) holds if and only if $\ell(\mathcal{H}_{\rho,d}) = \ell(\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d)$.

Recall that, for an integer $p \geq 2$, a partition $\lambda = (\lambda_1, \dots, \lambda_s)$ is said to be *p-restricted* if $\lambda_j - \lambda_{j+1} < p$ for $1 \leq j < s$ and $\lambda_s < p$. The simple \mathcal{H}_n -modules are parameterised by the *e-restricted* partitions of n , and for any $(\pi, l) \in \text{Bl}_e(n)$, the simple $\mathcal{H}_{\pi, l}$ -modules are parameterised by the *e-restricted* elements of $\text{Par}_e(\pi, l)$; see [10, Theorem 7.6] and [11, Theorem 4.13].

Suppose that A is a finite-dimensional algebra over a field k such that k is a splitting field for A , and let $m = \ell(A)$. If $\text{char } k = p > 0$, let \mathcal{RP}_k be the set of p -restricted partitions; if $\text{char } k = 0$, let \mathcal{RP}_k be the set of all partitions. Then a well-known argument shows that $\ell(A \wr \mathfrak{S}_d)$ is the number of tuples $(\lambda^1, \dots, \lambda^m)$ of elements of \mathcal{RP}_k such that $\sum_{j=1}^m |\lambda^j| = d$; see, for example, [25, Appendix]. The number of e -restricted partitions in $\text{Par}_e(\emptyset, 1)$ is equal to $e - 1$. Combining these facts, we see that $\ell(\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d)$ is the cardinality of the set Y_d of all tuples $(\lambda^1, \dots, \lambda^{e-1})$ of elements of \mathcal{RP}_F such that $\sum_{j=1}^{e-1} |\lambda^j| = d$.

Each partition $\lambda \in \text{Par}_e(\rho, d)$ corresponds to a tuple $(\lambda^{(0)}, \dots, \lambda^{(e-1)})$, where the definition of $\lambda^{(i)}$ is given before Lemma 4.13. It is easy to see that such a partition λ is e -restricted if and only if $\lambda^{(e-1)} = \emptyset$ (cf. [9, Lemma 4.1(1)]). Hence, $\ell(\mathcal{H}_{\rho,d}) = |X_d|$ where X_d is the set of all tuples $(\lambda^1, \dots, \lambda^{e-1})$ of partitions such that $\sum_i |\lambda^i| = d$. Observe that $Y_d \subset X_d$. Note that, for a prime p , all partitions of an integer n are p -restricted if and only if $n < p$. Hence, if (iii) holds, then $Y_d = X_d$ and so $\ell(\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d) = \ell(\mathcal{H}_{\rho,d})$; otherwise, $\ell(\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d) < \ell(\mathcal{H}_{\rho,d})$. This proves the equivalence between (ii) and (iii).

The last assertion of the proposition is an immediate consequence of Theorem 1.1. \square

9. ALTERNATIVE DESCRIPTIONS OF THE ISOMORPHISM

In this section, we assume all the notation and conventions of §6.3 and work again with the algebra $e_{\delta d} \overline{R}_{d\delta} e_{\delta d}$ for a fixed $d \geq 0$. We have constructed elements $\tau_r \in \sigma_r e_{\delta d} + \iota(S_{\{0\}}^{(r)})$ for $1 \leq r < d$ satisfying the relations of Theorem 6.14. We show that such elements τ_r are in some sense unique and use this fact to give alternative descriptions of τ_r .

9.1. A uniqueness result.

Lemma 9.1. *Let $1 \leq r < d$. Suppose that an element $\tau' \in \sigma_r e_{\delta d} + \iota(S_{\{0\}}^{(r)})$ satisfies the following property:*

- (i) *if $e > 2$, then $\eta_r(x)\tau' = \tau'\eta_{r+1}(x)$ for all $x \in (R_{\delta}^{\Lambda_0})_{\{0,1\}}$;*
- (ii) *if $e = 2$, then $(y_{2r} - y_{2r-1})\tau' = \tau'(y_{2(r+1)} - y_{2r+1})$.*

Then $\tau' = \tau_r$.

Proof. To simplify notation, we will assume that $r = 1$: the proof in the general case is obtained by a straightforward modification of the one below (cf. Remark 6.15). We write $\tau = \tau_1$. By the hypothesis and Theorem 6.14(iv), $\tau' = \tau + \iota(a)$ for some $a \in S_{\{0\}}^{(1)}$. Recall that we have an algebra isomorphism $\iota: \overline{R}_{\delta}^{\otimes d} \xrightarrow{\sim} \overline{R}_{\delta d}$, defined after Lemma 6.8.

In the case when $e = 2$, it follows from the hypothesis and Theorem 6.14(iv) that $(y_2 - y_1)\iota(a) = \iota(a)(y_3 - y_4)$. Moreover, we have $(R_{\delta}^{\Lambda_0})_{\{0\}} = F\{1\}$ by Proposition 5.4, whence $S_{\{0\}} = 1$ by Corollary 6.6, so a must be a scalar multiple of the identity. By the same Proposition and Lemma 6.5, we have $(\overline{y}_2 - \overline{y}_1)\overline{e}_{\delta} \neq 0$ in \overline{R}_{δ} , whence the elements $(y_2 - y_1)e_{\delta d} = \iota((\overline{y}_2 - \overline{y}_1)\overline{e}_{\delta} \otimes \overline{e}_{\delta}^{\otimes d-1})$ and $(y_4 - y_3)e_{\delta d} = \iota(\overline{e}_{\delta} \otimes (\overline{y}_2 - \overline{y}_1)\overline{e}_{\delta} \otimes \overline{e}_{\delta}^{\otimes d-2})$ are linearly independent. Hence, $a = 0$, so the lemma holds for $e = 2$.

Assuming that $e > 2$, note that a satisfies $\eta_1(x)\iota(a) = \iota(a)\eta_2(x)$ for all $x \in (R_{\delta}^{\Lambda_0})_{\{0,1\}}$. Using Corollary 6.6, we identify $(R_{\delta}^{\Lambda_0})_{\{0,1\}}$ with $(\overline{R}_{\delta})_{\{0,1\}}$. Then $a = \iota(a' \otimes \overline{e}_{\delta}^{\otimes d-2})$ for some $a' \in (R_{\delta}^{\Lambda_0})_{\{0\}} \otimes (R_{\delta}^{\Lambda_0})_{\{0\}}$. Further, we have $(x \otimes e_{\delta})a' = a'(e_{\delta} \otimes x)$ for all $x \in (R_{\delta}^{\Lambda_0})_{\{0,1\}}$. We will prove that $a' = 0$ (and hence $\tau = \tau'$) by considering $a'(e_{\delta} \otimes e(\mathbf{i}))$ for each $\mathbf{i} \in I^{\emptyset,1}$. First, consider the case when $i_e = i_{e-1} \pm 1$. Since $\deg(a) = 0$, we have $a'(e_{\delta} \otimes e(\mathbf{i})\psi_{e-1}) \in (R_{\delta}^{\Lambda_0}) \otimes (R_{\delta}^{\Lambda_0})_{\{1\}}$. On the other hand,

$$\begin{aligned} a'(e_{\delta} \otimes e(\mathbf{i})\psi_{e-1}) &= a'(e_{\delta} \otimes e(\mathbf{i})\psi_{e-1})(e_{\delta} \otimes e(s_{e-1}\mathbf{i})) \\ &= (e(\mathbf{i})\psi_{e-1} \otimes e_{\delta})a'(e_{\delta} \otimes e(s_{e-1}\mathbf{i})) \in R_{\delta}^{\Lambda_0} \otimes (R_{\delta}^{\Lambda_0})_{\{0\}}, \end{aligned}$$

whence $a'(e_\delta \otimes e(\mathbf{i})\psi_{e-1}) = 0$. Now, as $e > 2$, it follows easily from Proposition 5.4 that the linear map from $(R_\delta^{\Lambda_0})_{\{0\}}e(\mathbf{i})$ to $(R_\delta^{\Lambda_0})_{\{1\}}e(s_{e-1}\mathbf{i})$ given by $c \mapsto c\psi_{e-1}$ is injective. Hence, $a'(e_\delta \otimes e(\mathbf{i})) = 0$.

Finally, let $\mathbf{i} \in I^{\varnothing,1}$ be arbitrary. Using Lemma 5.1, one can easily show that there exists $\mathbf{j} \in I^{\varnothing,1}$ such that $j_e = i_e$ and $j_{e-1} \in \{j_e \pm 1\}$. By the case considered previously, $a'(e_\delta \otimes e(\mathbf{j})) = 0$. We have $\psi_{w_{\mathbf{i},\mathbf{j}}}\psi_{w_{\mathbf{j},\mathbf{i}}}e(\mathbf{i}) = e(\mathbf{i})$ (see the proof of Lemma 5.5), and hence

$$a'(e_\delta \otimes e(\mathbf{i})) = a'(e_\delta \otimes \psi_{w_{\mathbf{i},\mathbf{j}}}e(\mathbf{j})\psi_{w_{\mathbf{j},\mathbf{i}}}) = (\psi_{w_{\mathbf{i},\mathbf{j}}}e(\mathbf{j}) \otimes e_\delta)a'(e_\delta \otimes e(\mathbf{j}))(e_\delta \otimes \psi_{w_{\mathbf{j},\mathbf{i}}}e(\mathbf{i})) = 0. \quad \square$$

9.2. An explicit formula for τ_r . Using Lemma 9.1, we can give (without a complete proof) an explicit formula for $\tau = \tau_1$ when $d = 2$. For $e = 2$, we have already shown in the proof of Theorem 6.14(iv) that $\tau = (\sigma + 1)e(0101)$, where we set $\sigma := \sigma_1$. For arbitrary e ,

$$(9.1) \quad \tau = \sigma e_{\delta,\delta} + \sum_{\substack{\mathbf{i}, \mathbf{j} \in I^{\varnothing,1} \\ i_e = j_e}} (-1)^{i_e+1} \eta_1(\psi_{w_{\mathbf{j},\mathbf{i}}}) \eta_2(\psi_{w_{\mathbf{i},\mathbf{j}}}) e(\mathbf{i}\mathbf{j}),$$

where i_e is viewed as an element of \mathbb{Z} via the identification of I with $\{0, 1, \dots, e-1\}$. For example, for $e = 3$ we have

$$\tau = \sigma e_{\delta,\delta} - e(012012) + e(021021).$$

It is possible to show that the right-hand side τ' of (9.1) satisfies $\eta_1(x)\tau' = \tau'\eta_2(x)$ for all $x \in (R_\delta^{\Lambda_0})_{\{0,1\}}$ by technical calculations using the defining relations of $\overline{R}_{2\delta}$, and consequently that $\tau' = \tau$ by Lemma 9.1. These calculations are not included in the present paper, but equivalent calculations have been independently done by Kleshchev and Muth [22, Section 6] for KLR algebras of all untwisted affine ADE types (cf. Remark 3.10). For arbitrary r and d , we have $\tau_r = \zeta_{r,2}(\overline{e}_\delta^{\otimes r-1} \otimes \tau \otimes \overline{e}_\delta^{\otimes d-r-1})$ (cf. Remark 6.15).

9.3. A formula for τ_r via Hecke generators. Let $e \geq 2$ and $d \geq 1$, and assume all the notation and conventions introduced in §7.2 prior to Lemma 7.3. From now on, we identify H_{de}^Λ with R_{de}^Λ via the isomorphism \mathbf{BK}_{de}^Λ . As usual, for each $w \in \mathfrak{S}_{de}$ let $T_w = T_{j_1} \cdots T_{j_m} \in H_{de}^\Lambda$, where $w = s_{j_1} \cdots s_{j_m}$ is a reduced expression. Recall the elements $w_r \in \mathfrak{S}_{de}$ defined by (6.7).

Proposition 9.2. *We have $\tau_r = (-1)^e(e_{\delta^d} \overline{T}_{w_r} e_{\delta^d})_{\{0\}}$ whenever $1 \leq r < d$.*

It follows from (3.14) that $(e_{\delta^d} \overline{T}_{w_r} e_{\delta^d})_{\{0\}} = \zeta_{r,2}((e_{\delta,\delta} \overline{T}_{w_1} e_{\delta,\delta})_{\{0\}})$. Hence, the general case of Proposition 9.2 follows from the case when $d = 2$ and $r = 1$ (cf. Remark 6.15). We begin the proof of the proposition in that case with two lemmas.

Recall the power series $P_i, Q_i \in F[[y, y']]$ given by (3.16)–(3.19), and write $P_i^{(0)}$ and $Q_i^{(0)}$ for the constant coefficients of P_i and Q_i respectively. In particular, if $\xi = 1$, then

$$Q_i^{(0)} = \begin{cases} 1 & \text{if } i = 0, \\ 1 - i^{-1} & \text{if } i \notin \{0, 1, -1\}, \\ 2 & \text{if } e \neq 2 \text{ and } i = -1, \\ 1 & \text{if } e \neq 2 \text{ and } i = 1, \\ 1 & \text{if } e = 2 \text{ and } i = 1, \end{cases}$$

and if $\xi \neq 1$, then

$$Q_i^{(0)} = \begin{cases} 1 - \xi & \text{if } i = 0, \\ \xi(\xi^{i-1} - 1)/(\xi^i - 1) & \text{if } i \notin \{0, 1, -1\}, \\ \xi(\xi^{-2} - 1)/(\xi^{-1} - 1)^2 & \text{if } e \neq 2 \text{ and } i = -1, \\ 1 & \text{if } e \neq 2 \text{ and } i = 1, \\ 1/(\xi - 1) & \text{if } e = 2 \text{ and } i = 1. \end{cases}$$

One easily checks (using the fact that e is prime if $\xi = 1$) that

$$(9.2) \quad Q_0^{(0)} Q_1^{(0)} \cdots Q_{e-1}^{(0)} = -1.$$

Lemma 9.3. *Assume that $d = 2$. Then $(-1)^e (e_{\delta, \delta} \bar{T}_{w_1} e_{\delta, \delta})_{\{0\}} \in \sigma_1 e_{\delta, \delta} + \iota(S_{\{0\}})$.*

Proof. Let $\mathbf{j}^{(1)}, \mathbf{j}^{(2)} \in I^{\emptyset, 1}$, and let $w_1 = s_{t_{e_2}} \cdots s_{t_2} s_{t_1}$ be a reduced expression for w_1 , so that

$$(9.3) \quad (e_{\delta, \delta} \bar{T}_{w_1} e(\mathbf{j}^{(1)} \mathbf{j}^{(2)}))_{\{0\}} = (e_{\delta, \delta} \bar{T}_{i_{e_2}} \cdots \bar{T}_{i_1} e(\mathbf{j}^{(1)} \mathbf{j}^{(2)}))_{\{0\}}.$$

Applying (3.14) to the terms $\bar{T}_{i_1}, \dots, \bar{T}_{i_{e_2}}$ in this order and using the relations (3.2) and (3.5), we decompose the right-hand side of (9.3) as a sum of 2^{e^2} terms; these terms correspond to the choice of either the first or the second summand of (3.14) at each of the e^2 steps. The term corresponding to choosing the first summand in every case is $a_{\{0\}}$, where $a = a_{e^2} \cdots a_1$, $a_k = \psi_{t_k} Q_{i_{t_k}^{(k)} - i_{t_k+1}^{(k)}}(y_{i_{t_k}^{(k)}}, y_{i_{t_k+1}^{(k)}})$ for each k , and $\mathbf{i}^{(k)} = (i_1^{(k)}, \dots, i_{2e}^{(k)}) = s_{t_{k-1}} \cdots s_{t_1}(\mathbf{j}^{(1)} \mathbf{j}^{(2)})$. By Theorem 3.5(ii), the sum b of the other $2^{e^2} - 1$ terms belongs to $(\sum_{w \in \mathfrak{S}_{2e}, w < w_1} e_{\delta, \delta} \psi_w \bar{R}_{\delta, \delta})_{\{0\}} \subset (\bar{R}_{\delta, \delta})_{\{0\}} = \iota(S_{\{0\}})$, where the containment follows from the claim in the proof of Lemma 6.9.

Using Lemma 6.9(ii), since each of $\mathbf{j}^{(1)}, \mathbf{j}^{(2)}$ is a permutation of $(0, \dots, e-1)$, we have

$$a_{\{0\}} = \left(\prod_{i, j \in I} Q_{i-j}^{(0)} \right) \psi_{w_1} e(\mathbf{j}^{(1)} \mathbf{j}^{(2)}) = (-1)^e \sigma_1 e(\mathbf{j}^{(1)} \mathbf{j}^{(2)})$$

by (9.2) and the definition of σ_1 . Therefore, $(e_{\delta, \delta} \bar{T}_{w_1} e(\mathbf{j}^{(1)} \mathbf{j}^{(2)}))_{\{0\}} = a_{\{0\}} + b \in (-1)^e \sigma_1 e_{\delta, \delta} + \iota(S_{\{0\}})$, and the lemma follows by summing over all $\mathbf{j}^{(1)}, \mathbf{j}^{(2)} \in I^{\emptyset, 1}$. \square

Lemma 9.4. *Let $1 \leq r \leq d$. The element $\bar{X}_{(r-1)e+1} e_{\delta d}$ of $\bar{R}_{d\delta}^\Lambda$ commutes with $e_{\delta d}$ and with each of $\bar{T}_j e_{\delta d}$ whenever $(r-1)e+1 \leq j < de$. Also, $\bar{T}_{(r-1)e+t}$ commutes with $e_{\delta d}$ if $1 \leq t < e$.*

Proof. By (3.15), the element $\bar{X}_{(r-1)e+1}$ belongs to the subalgebra of \bar{R}_{de}^Λ generated by $\{\overline{e(\mathbf{i})} \mid \mathbf{i} \in I^{d\delta}\} \cup \{\bar{y}_{(r-1)e+1}\}$, and each \bar{T}_l ($1 \leq l < de$) belongs to the subalgebra generated by $\{\overline{e(\mathbf{i})} \mid \mathbf{i} \in I^{d\delta}\} \cup \{\bar{\psi}_l, \bar{y}_l, \bar{y}_{l+1}\}$. Hence, each of the elements $\bar{X}_{(r-1)e+1}$ and $\bar{T}_{(r-1)e+t}$ ($1 \leq t < e$) commutes with $e_{\delta d}$.

By the defining relations of H_{de}^Λ , it is clear that $\bar{X}_{(r-1)e+1}$ commutes with \bar{T}_j for $(r-1)e+1 < j < de$. Thus, it only remains to show that $\bar{X}_{(r-1)e+1} \bar{T}_{(r-1)e+1} e_{\delta d} = \bar{T}_{(r-1)e+1} \bar{X}_{(r-1)e+1} e_{\delta d}$. By the defining relations (3.2)–(3.6), $\bar{X}_{(r-1)e+1} e_{\delta d}$ commutes with $y_{(r-1)e+1}$, $y_{(r-1)e+2}$ and the elements $\overline{e(\mathbf{i})}$ for $\mathbf{i} \in I^{d\delta}$. Moreover, we have $\bar{\psi}_{(r-1)e+1} e_{\delta d} = 0$ due to Lemma 4.5 and the statement after Lemma 6.8. By an observation in the previous paragraph, the required identity follows. \square

Proof of Proposition 9.2. By the discussion following the statement of the proposition, we may (and do) assume that $d = 2$ and $r = 1$. Whenever $1 \leq t \leq 2$, define elements $\tilde{X}_{t,l} \in \bar{R}_{d\delta}^\Lambda$ for $1 \leq l \leq e$ as follows:

- (i) if $\xi = 1$, then $\tilde{X}_{t,1} = 0$ and $\tilde{X}_{t,l+1} = \bar{T}_{(t-1)e+l} \tilde{X}_{t,l} \bar{T}_{(t-1)e+l} + \bar{T}_{(t-1)e+l}$ for $1 \leq l < e$;
- (ii) if $\xi \neq 1$, then $\tilde{X}_{t,1} = 1$ and $\tilde{X}_{t,l+1} = \xi^{-1} \bar{T}_{(t-1)e+l} \tilde{X}_{t,l} \bar{T}_{(t-1)e+l}$ for $1 \leq l < e$.

We claim that

$$(9.4) \quad \bar{X}_{(t-1)e+l} e_{\delta, \delta} = (\tilde{X}_{t,l} + \bar{X}_{(t-1)e+1}) e_{\delta, \delta} \quad \text{if } \xi = 1,$$

$$(9.5) \quad \bar{X}_{(t-1)e+l} e_{\delta, \delta} = \tilde{X}_{t,l} \bar{X}_{(t-1)e+1} e_{\delta, \delta} \quad \text{if } \xi \neq 1$$

for $1 \leq l \leq e$ and $1 \leq t \leq 2$. These equalities can be proved by induction on l : the base case $l = 1$ is clear, and, for $\xi \neq 1$, the inductive step $l \rightarrow l+1$ is proved, using the defining relations of H_{2e}^Λ and Lemma 9.4 as follows:

$$\begin{aligned} \overline{X}_{(t-1)e+l+1}e_{\delta,\delta} &= \xi^{-1}\overline{T}_{(t-1)e+l}\overline{X}_{(t-1)e+l}\overline{T}_{(t-1)e+l}e_{\delta,\delta} \\ &= \xi^{-1}\overline{T}_{(t-1)e+l}\tilde{X}_{t,l}\overline{X}_{(t-1)e+1}e_{\delta,\delta}\overline{T}_{(t-1)e+l}e_{\delta,\delta} \\ &= \xi^{-1}\overline{T}_{(t-1)e+l}\tilde{X}_{t,l}\overline{T}_{(t-1)e+l}\overline{X}_{(t-1)e+1}e_{\delta,\delta} \\ &= \tilde{X}_{t,l+1}\overline{X}_{(t-1)e+1}e_{\delta,\delta}. \end{aligned}$$

The proof of the inductive step for $\xi = 1$ is similar and is left as an exercise.

Let $t \in \{1, 2\}$. It follows from Lemma 9.4 that $\tilde{X}_{t,l}$ commutes with $\overline{X}_{(t-1)e+1}e_{\delta,\delta}$ for $1 \leq l \leq e$. Using this observation, Equations (9.4)–(9.5) and the fact that $(\overline{X}_{(t-1)e+1} - \hat{0})e_{\delta,\delta}$ is nilpotent for $t = 1, 2$ (cf. Theorem 3.2(ii) and (3.13)), we see that

$$(9.6) \quad \eta_t(e(\mathbf{i}))\overline{R}_{d\delta}^\Lambda e_{\delta,\delta} = \{v \in \overline{R}_{d\delta}^\Lambda e_{\delta,\delta} \mid (\tilde{X}_{t,l} - \hat{i}_l)^L v = 0 \text{ for } L \gg 0 \text{ and all } l = 1, \dots, e\}$$

whenever $\mathbf{i} \in I^{\varnothing,1}$. For $1 \leq l < e$, we have

$$(9.7) \quad T_l T_{w_1} = T_{s_l w_1} = T_{w_1 s_{e+l}} = T_{w_1} T_{e+l}$$

since $\ell(s_l w_1) = \ell(w_1) + 1$. It follows easily by an inductive argument that

$$(9.8) \quad \tilde{X}_{1,l}\overline{T}_{w_1} = \overline{T}_{w_1}\tilde{X}_{2,l}$$

for all $l = 1, \dots, e$. By (9.6) and (9.8), we have $e_{\delta,\delta}\overline{T}_{w_1}\eta_2(e(\mathbf{i}))\overline{R}_{d\delta}^\Lambda e_{\delta,\delta} \subset \eta_1(e(\mathbf{i}))\overline{R}_{d\delta}^\Lambda e_{\delta,\delta}$, whence $e_{\delta,\delta}(\overline{T}_{w_1})_{\{0\}}\eta_2(e(\mathbf{i})) = \eta_1(e(\mathbf{i}))(\overline{T}_{w_1})_{\{0\}}\eta_2(e(\mathbf{i}))$ for all $\mathbf{i} \in I^{\varnothing,1}$. A similar argument (with (9.6) replaced by an analogous statement where $e_{\delta,\delta}\overline{R}_{d\delta}^\Lambda$ is viewed as a right module over $\overline{R}_{d\delta}$) establishes the first equality in the following equation:

$$(9.9) \quad \eta_1(e(\mathbf{i}))(\overline{T}_{w_1})_{\{0\}}e_{\delta,\delta} = \eta_1(e(\mathbf{i}))(\overline{T}_{w_1})_{\{0\}}\eta_2(e(\mathbf{i})) = e_{\delta,\delta}(\overline{T}_{w_1})_{\{0\}}\eta_2(e(\mathbf{i})).$$

Assume first that $e > 2$. By (3.14), we have

$$(9.10) \quad \psi_{(t-1)e+l}\eta_t(e(\mathbf{i})) = ((\overline{T}_{(t-1)e+l})_{\{0,1\}} + P_{i_l-i_{l+1}}^{(0)})(Q_{i_l-i_{l+1}}^{(0)})^{-1}\eta_t(e(\mathbf{i}))$$

whenever $1 \leq l < e$ since the left-hand side is homogeneous of degree 0 or 1. Let $\tau' = (-1)^e(e_{\delta,\delta}\overline{T}_{w_1}e_{\delta,\delta})_{\{0\}} \in (\overline{R}_{d\delta}^\Lambda)_{\{0\}} = (\overline{R}_{d\delta})_{\{0\}}$. Recall that $e_{\delta,\delta}\overline{R}_{d\delta}^\Lambda e_{\delta,\delta}$ is nonnegatively graded by Lemma 6.9(iii). It follows from (9.9) that $\eta_1(e(\mathbf{i}))\tau' = \tau'\eta_2(e(\mathbf{i}))$ for all $\mathbf{i} \in I^{\varnothing,1}$. By (9.10), (9.7) and degree considerations, we have $\eta_1(\psi_l e(\mathbf{i}))\tau' = \tau'\eta_2(\psi_l e(\mathbf{i}))$ whenever $1 \leq l < e$ and $\mathbf{i} \in I^{\varnothing,1}$. By Lemmas 9.1 and 9.3 together with the fact that $(R_\delta^{\Lambda_0})_{\{0,1\}}$ is contained in the subalgebra generated by the elements of the form $e(\mathbf{i})$ and $\psi_l e(\mathbf{i})$ for $e(\mathbf{i}) \in I^{\varnothing,1}$ and $1 \leq l < e$, we have $\tau_1 = \tau'$.

Finally, consider the case when $e = 2$. By Lemma 7.3, we have

$$(y_{2t} - y_{2t-1})e_{\delta,\delta} = \begin{cases} ((\overline{T}_{2t-1})_{\{0,1,2\}} + 1)e_{\delta,\delta} & \text{if } \text{char } F = 2, \\ (2(\overline{T}_{2t-1})_{\{0,1,2\}} + 1)e_{\delta,\delta} & \text{if } \text{char } F \neq 2. \end{cases}$$

for $t = 1, 2$. Using this formula, Equation (9.7), Lemma 9.4 and degree considerations, we obtain $(y_2 - y_1)(e_{\delta,\delta}\overline{T}_{w_1}e_{\delta,\delta})_{\{0\}} = (e_{\delta,\delta}\overline{T}_{w_1}e_{\delta,\delta})_{\{0\}}(y_4 - y_3)$, and hence $\tau_1 = (e_{\delta,\delta}\overline{T}_{w_1}e_{\delta,\delta})_{\{0\}}$ by Lemmas 9.1 and 9.3. \square

Let $\mathcal{H}_{\rho,d}$ be a RoCK block of residue κ . We identify $R_{|\rho|+de}^{\Lambda_0}$ with $\mathcal{H}_{|\rho|+de}$ via the isomorphism $\text{BK}_{|\rho|+de}$. Thus, $\mathcal{H}_{|\rho|+de}$ becomes a graded algebra, and $C = C_{\rho,d}$ becomes a graded subalgebra of $\mathcal{H}_{|\rho|+de}$. If $1 \leq r < d$, define $w'_r = \prod_{j=1}^e (|\rho| + (r-1)e + j, |\rho| + re + j) \in \mathfrak{S}_{|\rho|+de}$ (cf. (6.7)).

Proposition 9.5. *The restriction of the graded algebra isomorphism $\Xi: \mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d \xrightarrow{\sim} C$ defined in Section 7 to $F\mathfrak{S}_d$ may be described as follows: $\Xi(s_r) = (-1)^e (f_{\rho,d} T_{w_r'} f_{\rho,d})_{\{0\}}$ whenever $1 \leq r < d$.*

Proof. We may assume that the integer N determining Λ in §7.2 is chosen to be large enough so that $R_{|\rho|+de}^{\Lambda_0} = (R_{|\rho|+de}^{\Lambda_0})_{\leq 2N-2}$. By definition, $\Xi(s_r) = \Omega(\text{rot}_\kappa(\tau_r))$. We define $\widehat{R}_{d\delta}^\Lambda$ to be the quotient of $R_{d\delta}^\Lambda$ by the two-sided ideal generated by $\{e(\mathbf{i}) \mid \mathbf{i} \in I^{d\delta} \setminus \mathcal{E}_{d,\kappa}\}$ and identify $(\widehat{R}_{d\delta})_{\{0,1,2\}}$ with $(\widehat{R}_{d\delta}^\Lambda)_{\{0,1,2\}}$ (cf. §7.2). It follows from (3.14) that the automorphism of $R_{d\delta}^\Lambda$ given by (3.25) fixes $T_k e_{d\delta}$ whenever $1 \leq k < de$, so by Proposition 9.2 we have $\text{rot}_\kappa(\tau_r) = (-1)^e \text{rot}_\kappa((e_{\delta d} \overline{T}_{w_r} e_{\delta d})_{\{0\}}) = (-1)^e e_{\delta d} (\widehat{T}_{w_r})_{\{0\}} e_{\delta d}$, where $\widehat{\cdot}: R_{d\delta} \rightarrow \widehat{R}_{d\delta}$ is the natural projection. By the choice of N , the map $\omega: R_{d\delta} \rightarrow R_{\text{cont}(\rho),d\delta}^{\Lambda_0}$ of Proposition 4.11 induces a homomorphism $\omega^\Lambda: R_{d\delta}^\Lambda \rightarrow R_{\text{cont}(\rho),d\delta}^{\Lambda_0}$. It follows from (3.14) that $\omega^\Lambda(T_k e_{d\delta}) = \iota_{|\rho|}^{|\rho|+de}(e_{\text{cont}(\rho)}) T_{|\rho|+k e_{\text{cont}(\rho)}+d\delta}$ whenever $1 \leq k < de$. Therefore, $\omega^\Lambda(T_{w_r} e_{d\delta}) = \iota_{|\rho|}^{|\rho|+de}(e_{\text{cont}(\rho)}) T_{w_r'} e_{\text{cont}(\rho)+d\delta}$, and so, as $\Omega(\widehat{e}_{\delta d}) = f_{\rho,d}$, we have $\Omega(e_{\delta d} (\widehat{T}_{w_r})_{\{0\}} e_{\delta d}) = (f_{\rho,d} T_{w_r'} f_{\rho,d})_{\{0\}}$. \square

Theorem 9.6. *Suppose that $\xi = 1$, and let $f = f_{\rho,d}$. Then we have a graded algebra isomorphism $\mathcal{H}_{\rho,0} \otimes (\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d) \xrightarrow{\sim} f\mathcal{H}_{|\rho|+de}f$ given as follows:*

$$\begin{aligned} a \otimes (b_{\emptyset,1}^{\otimes r-1} \otimes T_l b_{\emptyset,1} \otimes b_{\emptyset,1}^{\otimes d-r}) &\mapsto a T_{|\rho|+(r-1)e+lf} & \text{for } 1 \leq r \leq d, 1 \leq l < e, \\ a \otimes s_r &\mapsto a(f T_{w_r'} f)_{\{0\}} & \text{for } 1 \leq r < d \end{aligned}$$

for all $a \in \mathcal{H}_{\rho,0}$.

Proof. Due to (4.8) and Proposition 3.3, it is enough to show that we have an isomorphism from $\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d$ onto C given by

$$(9.11) \quad b_{\emptyset,1}^{\otimes r-1} \otimes T_l b_{\emptyset,1} \otimes b_{\emptyset,1}^{\otimes d-r} \mapsto T_{|\rho|+(r-1)e+lf} \quad \text{for } 1 \leq r \leq d, 1 \leq l < e,$$

$$(9.12) \quad s_r \mapsto (f T_{w_r'} f)_{\{0\}} \quad \text{for } 1 \leq r < d.$$

We identify $\mathcal{H}_{\emptyset,1}$ with $R_\delta^{\Lambda_0}$ via BK_e , so that $b_{\emptyset,1} = e_\delta$. For $1 \leq r \leq d, 1 \leq l < e$ and $\mathbf{i} \in I^\delta$, we have

$$\begin{aligned} \Xi(e_\delta^{\otimes r-1} \otimes e(\mathbf{i}) \otimes e_\delta^{\otimes d-r}) &= \sum e(\mathbf{j} \mathbf{i}^{(1)} \dots \mathbf{i}^{(r-1)} (\mathbf{i}^{+\kappa}) \mathbf{i}^{(r+1)} \dots \mathbf{i}^{(d)}), \\ \Xi(e_\delta^{\otimes r-1} \otimes \psi_l e_\delta \otimes e_\delta^{\otimes d-r}) &= \psi_{|\rho|+(r-1)e+l} \sum e(\mathbf{j} \mathbf{i}^{(1)} \dots \mathbf{i}^{(r-1)} (\mathbf{i}^{+\kappa}) \mathbf{i}^{(r+1)} \dots \mathbf{i}^{(d)}), \\ \Xi(e_\delta^{\otimes r-1} \otimes (y_l - y_{l+1}) e_\delta \otimes e_\delta^{\otimes d-r}) &= \\ &= (y_{|\rho|+(r-1)e+l} - y_{|\rho|+(r-1)e+l+1}) \sum e(\mathbf{j} \mathbf{i}^{(1)} \dots \mathbf{i}^{(r-1)} (\mathbf{i}^{+\kappa}) \mathbf{i}^{(r+1)} \dots \mathbf{i}^{(d)}), \end{aligned}$$

where each sum is over all $\mathbf{j} \in I^{\rho,0}$ and $\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(r-1)}, \mathbf{i}^{(r+1)}, \dots, \mathbf{i}^{(d)} \in I_{+\kappa}^{\emptyset,1}$. Hence, by Theorem 3.2(i)(iii), we have $\Xi(b_{\emptyset,1}^{\otimes(r-1)} \otimes T_l b_{\emptyset,1} \otimes b_{\emptyset,1}^{\otimes(d-r)}) = T_{|\rho|+(r-1)e+lf}$. By Proposition 9.5, it follows that the composition of Ξ with the automorphism of $\mathcal{H}_{\emptyset,1} \wr \mathfrak{S}_d$ that sends s_r to $(-1)^e s_r$ for all r and is the identity on $\mathcal{H}_{\emptyset,1}^{\otimes d}$ is given by (9.11)–(9.12). \square

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